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Real-time Height-field Simulation of Sand and Water Mixtures

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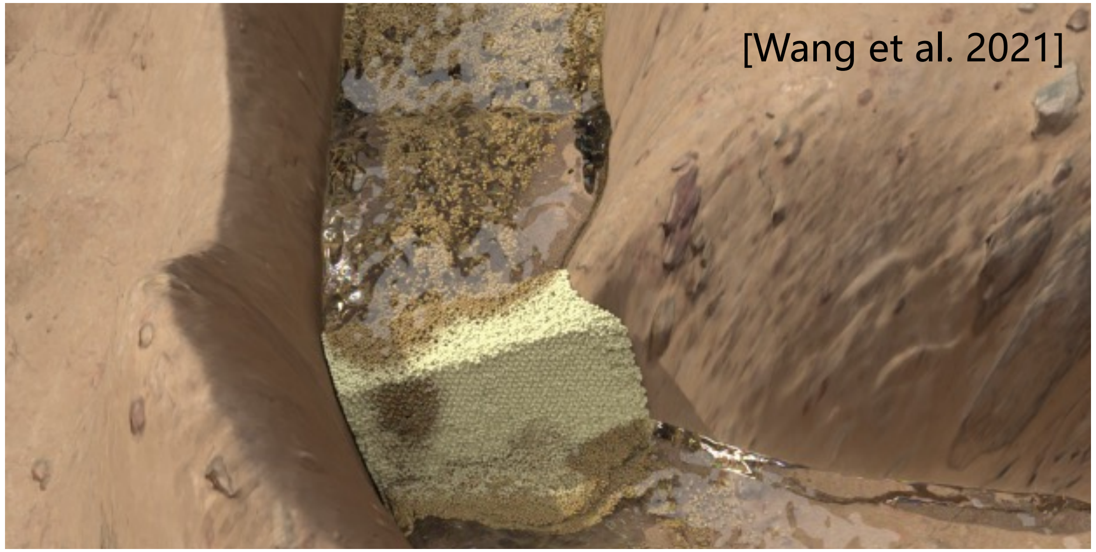
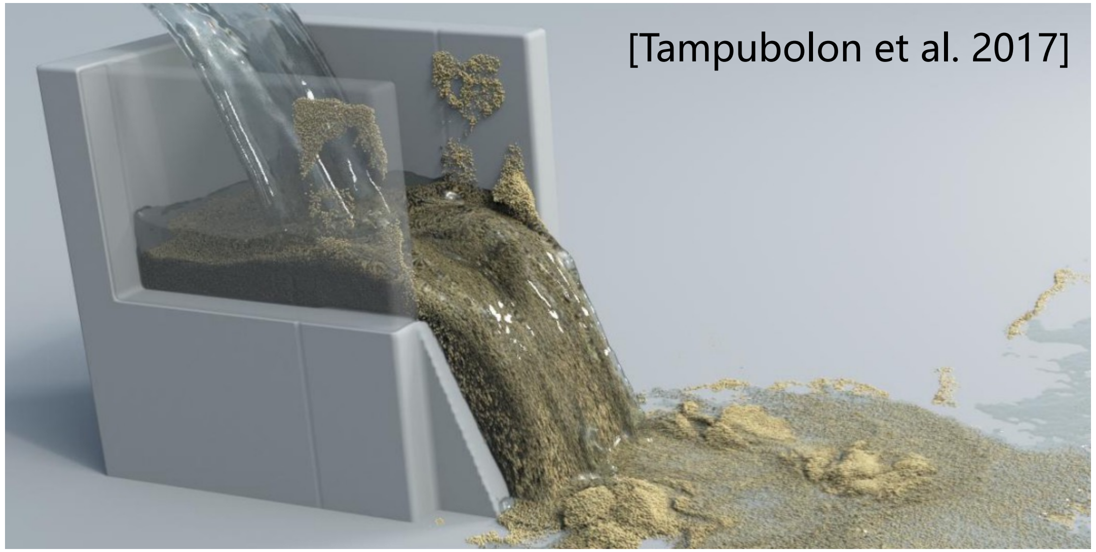
Dry Sand
Water
$\frac{\partial u}{\partial t} + u \cdot \nabla u = g - \frac{1}{\rho} \nabla p + \upsilon \nabla \cdot \nabla u$
$\nabla \cdot u = 0$
Wet Sand
Water-Sand Coupling
Elastoplasticity
Saturation

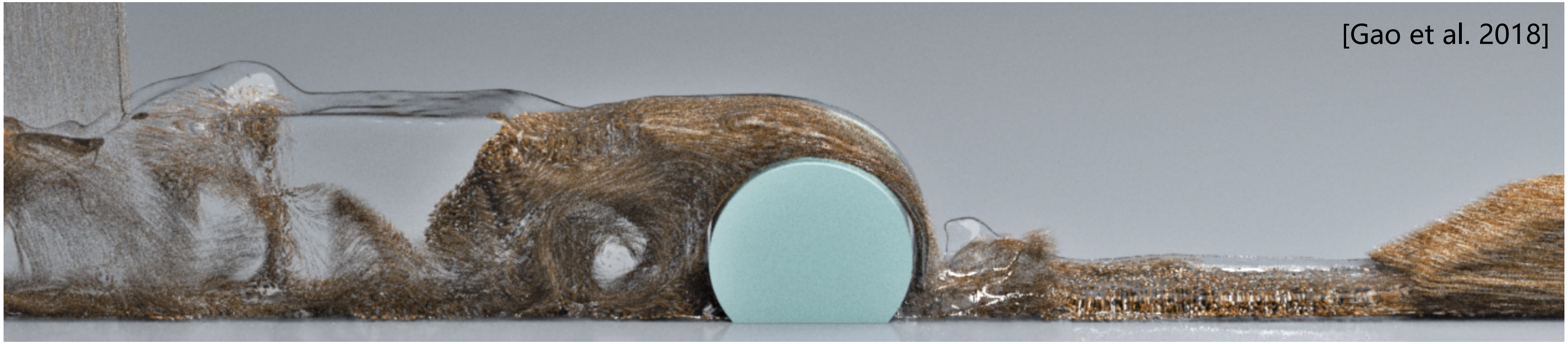
Prior works



[Wang et al. 2021]



[Tampubolon et al. 2017]



[Gao et al. 2018]

Prior works

Particle-based methods, like MPM, SPH, and DEM

- ✓ Lots of details
- ✓ High fidelity
- ✗ Slow
- ✗ Not suitable for real-time applications

Real-time Sand and Water Mixtures

1024 x 1024 Grid

2 ms/step

RTX 4080

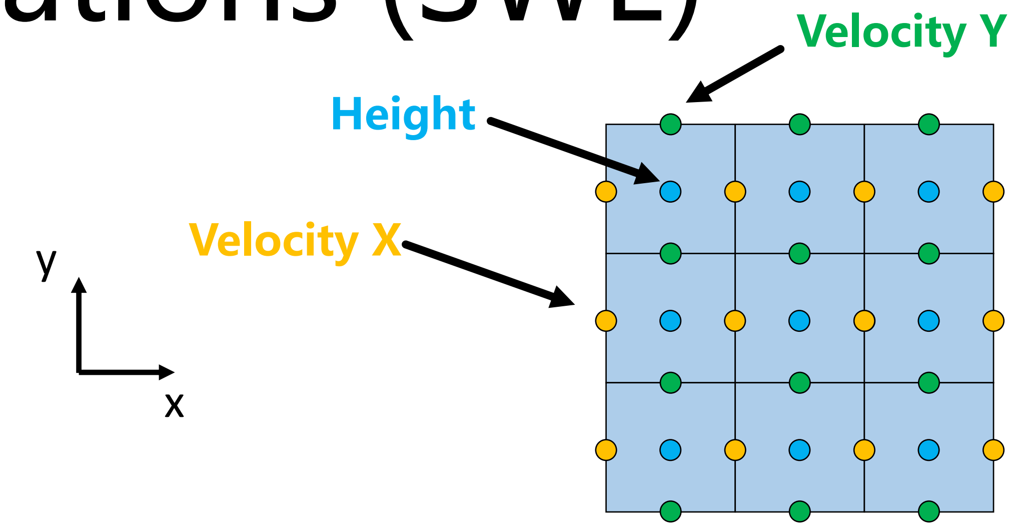
Shallow Water Equations (SWE)

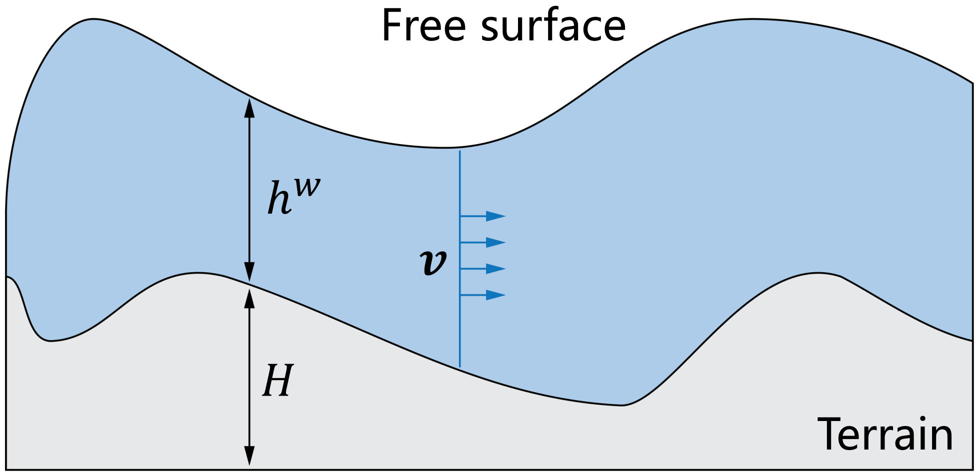
- Height integration

$$\frac{Dh^w}{Dt} = -h^w \nabla \cdot v^w$$

- Velocity integration

$$\frac{Dv^w}{Dt} = -g\nabla(H + h^w)$$





Shallow Water Equations (SWE)

- Height integration

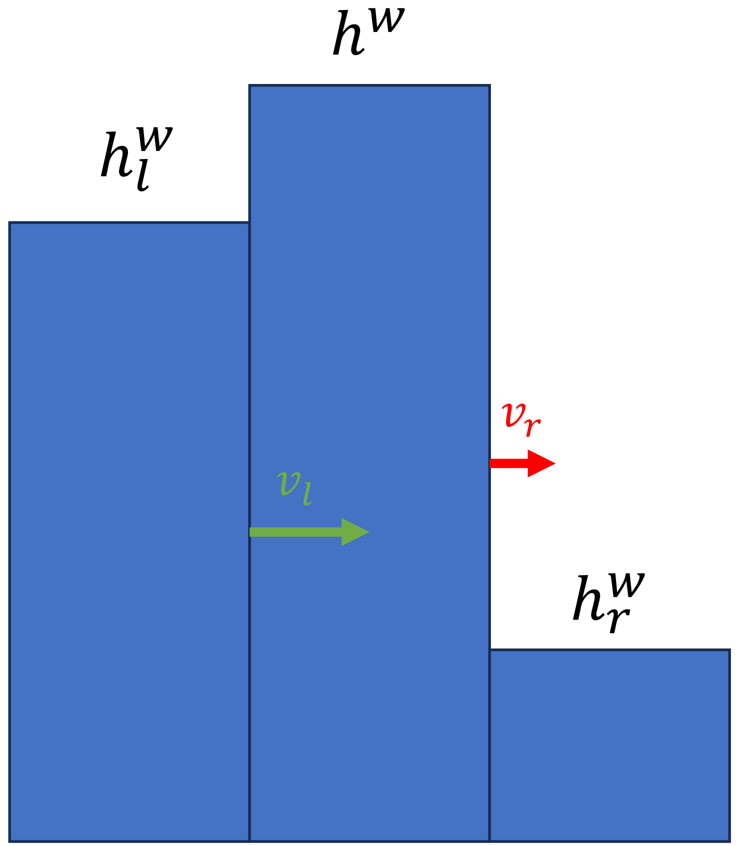
$$\frac{Dh^w}{Dt} = -h^w \nabla \cdot v^w$$

- Rewrite

$$\frac{\partial h^w}{\partial t} = -\nabla \cdot (h^w v^w)$$

- Discretize

$$\Delta h^w = -\nabla \cdot (h^w v^w)\Delta t$$
$$= \frac{\Delta t(h^w v^w)_l - \Delta t(h^w v^w)_r}{\Delta x}$$



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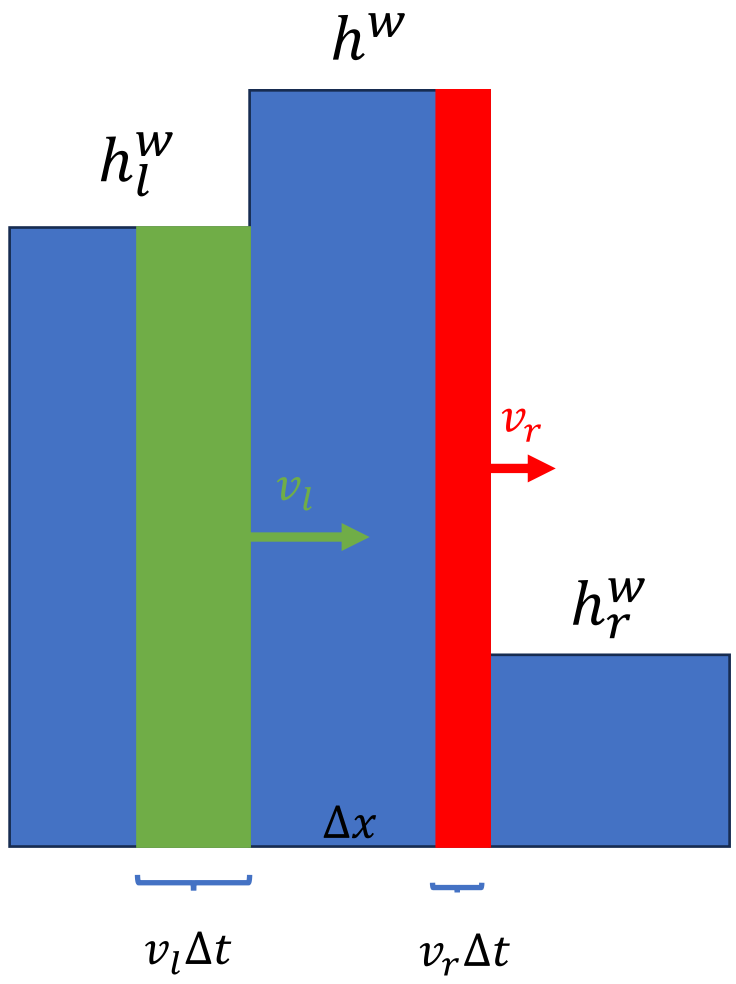


Shallow Water Equations (SWE)

$$\frac{\partial v^w}{\partial t} + v^w \cdot \nabla v^w = \cancel{g} - \frac{1}{\rho}\nabla p + \cancel{v^w \nabla \cdot \nabla v^w}$$

$$p = \rho g(h + H)$$

$$\frac{Dv^w}{Dt} = -g\nabla(H + h^w)$$

Shallow Water Equations (SWE)

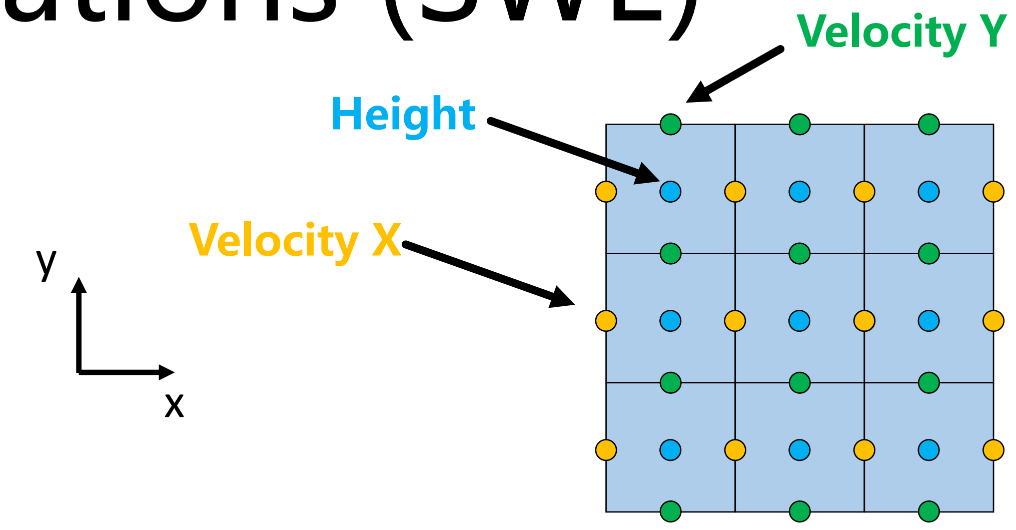
- Height integration

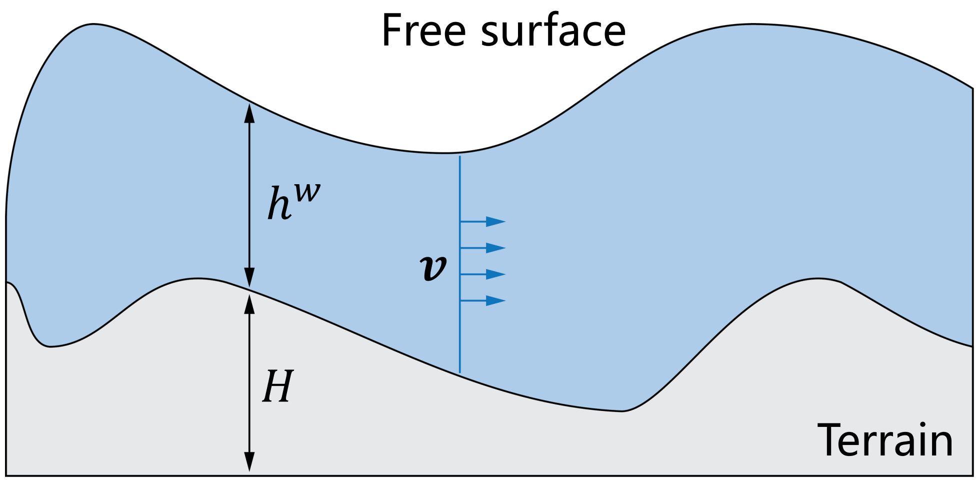
$$\frac{Dh^w}{Dt} = -h^w \nabla \cdot v^w$$

- Velocity integration

$$\frac{Dv^w}{Dt} = -g\nabla(H + h^w)$$

✓ 2.5D representation
✓ Super fast





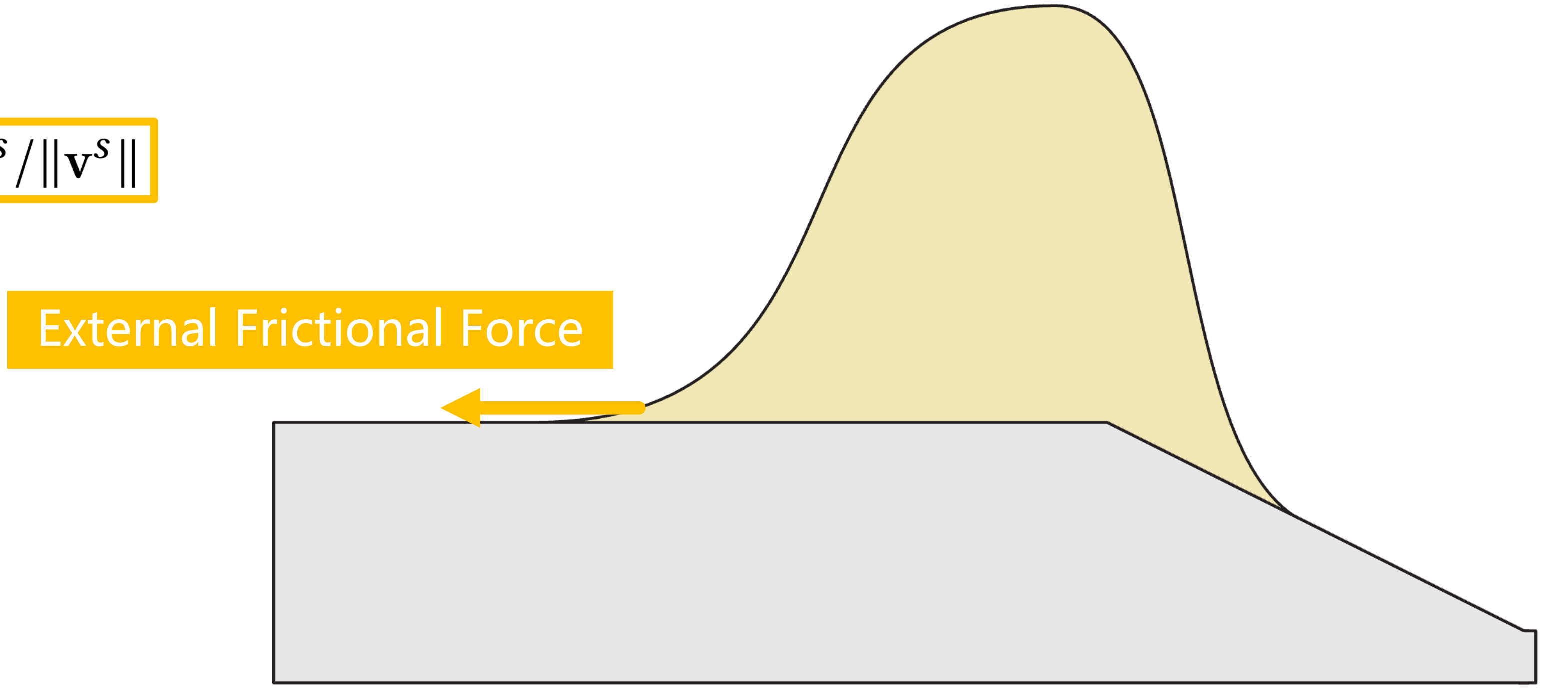
Shallow Sand Equations (SSE) [Zhu et al. 2021]

- Height integration

$$\frac{Dh^s}{Dt} = -h^s \nabla \cdot \mathbf{v}^s$$

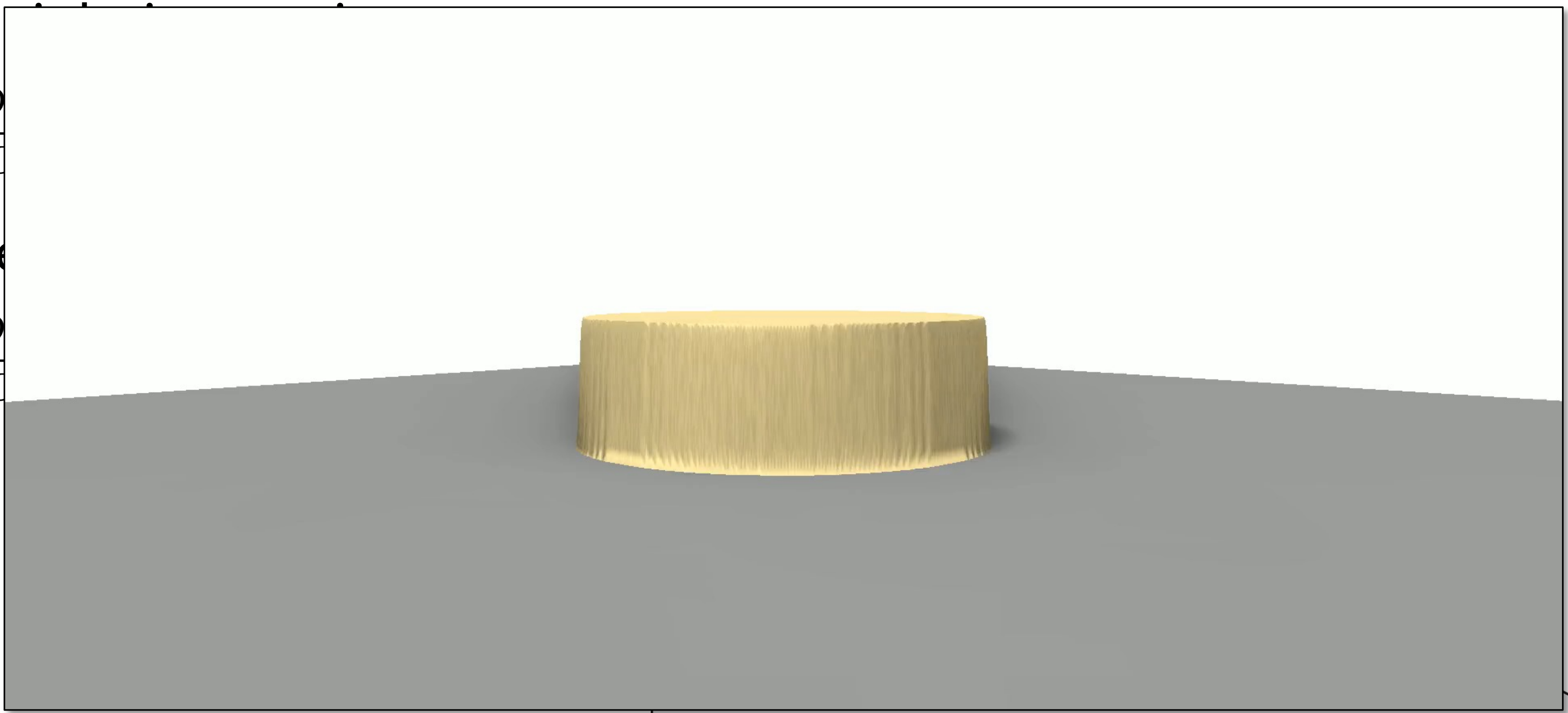
- Velocity integration

$$\frac{D\mathbf{v}^s}{Dt} = -g\nabla(H + h^s) \boxed{-\mu g\, \mathbf{v}^s / \|\mathbf{v}^s\|}$$



Shallow Sand Equations (SSE) [Zhu et al. 2021]

- H
- Ve



Shallow Sand Equations (SSE)

[Tampubolon et al. 2017]

- Height integration

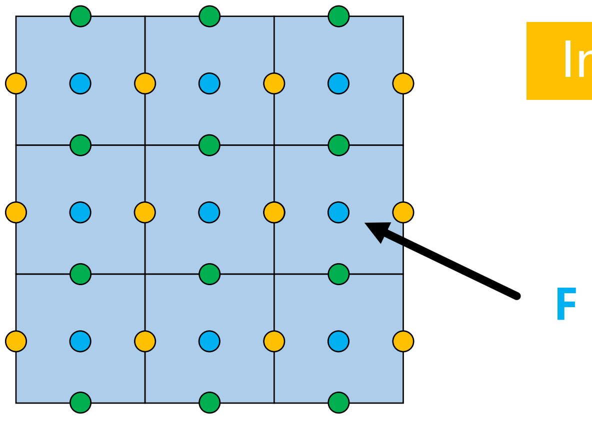
$$\frac{Dh^s}{Dt} = -h^s \nabla \cdot \mathbf{v}^s$$

- Velocity integration

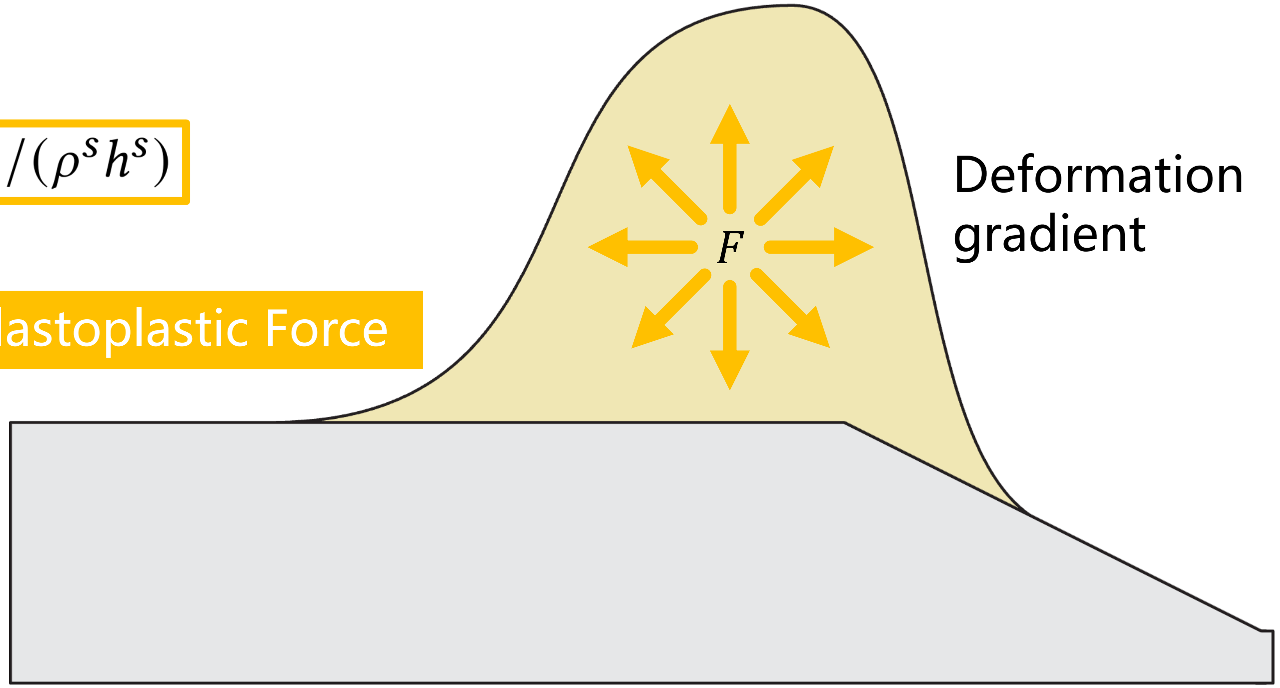
$$\frac{D\mathbf{v}^s}{Dt} = -g\nabla(H + h^s) - \mu g\, \mathbf{v}^s / \|\mathbf{v}^s\| \boxed{+ \mathbf{f}_{\mathcal{I}} / (\rho^s h^s)}$$

$$\boldsymbol{\sigma} = \frac{1}{\det(\mathbf{F})} \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T \qquad c_f \mathrm{tr}(\boldsymbol{\sigma}) + \left\| \boldsymbol{\sigma} - \frac{\mathrm{tr}(\boldsymbol{\sigma})}{d} \mathbf{I} \right\| \leq c_c(\phi)$$

$$\mathbf{f}_{\mathcal{I}} = \left(\frac{\partial(h^s \boldsymbol{\sigma}_{xx})}{\partial x} + \frac{\partial(h^s \boldsymbol{\sigma}_{xy})}{\partial y}, \frac{\partial(h^s \boldsymbol{\sigma}_{yx})}{\partial x} + \frac{\partial(h^s \boldsymbol{\sigma}_{yy})}{\partial y} \right)$$



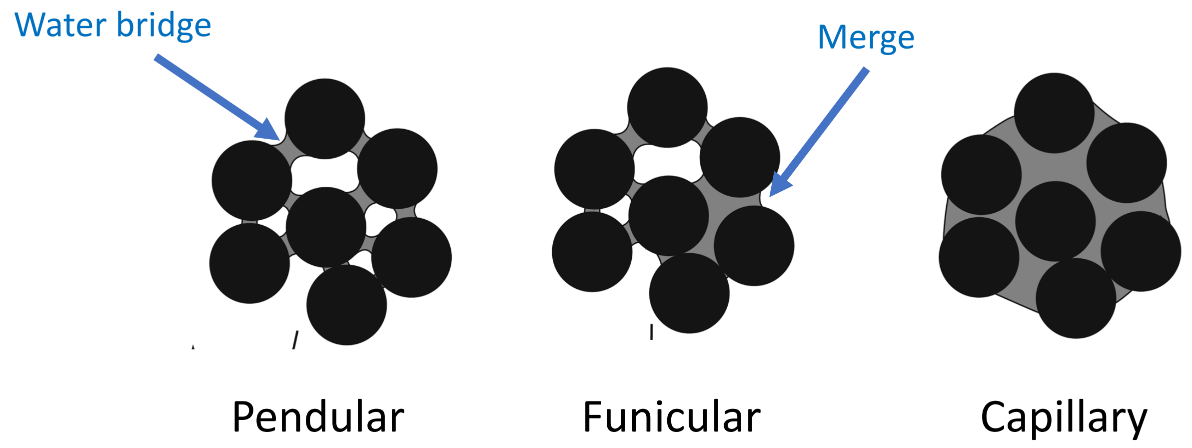
Internal Elastoplastic Force



Saturation States of Wet Sand

- Modified Drucker-Prager yielding condition

$$c_{\mathrm{f}}\mathrm{tr}(\boldsymbol{\sigma}) + \left\| \boldsymbol{\sigma} - \frac{\mathrm{tr}(\boldsymbol{\sigma})}{d}\mathbf{I} \right\| \leq c_{\mathrm{c}}(\phi)$$



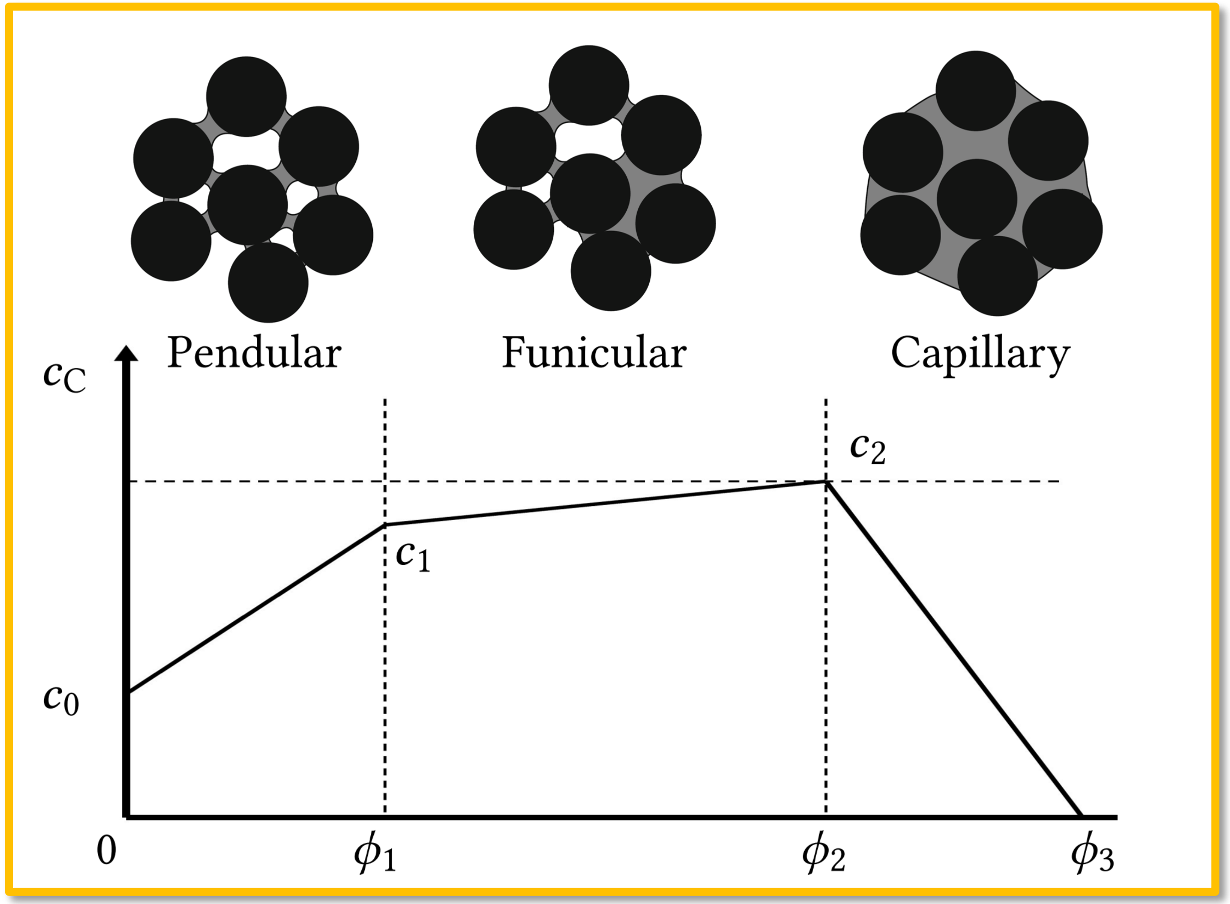
Saturation States of Wet Sand

- Modified Drucker-Prager yielding condition

$$c_{\mathrm{f}}\mathrm{tr}(\boldsymbol{\sigma}) + \left\| \boldsymbol{\sigma} - \frac{\mathrm{tr}(\boldsymbol{\sigma})}{d}\mathbf{I} \right\| \leq c_{\mathrm{c}}(\phi)$$

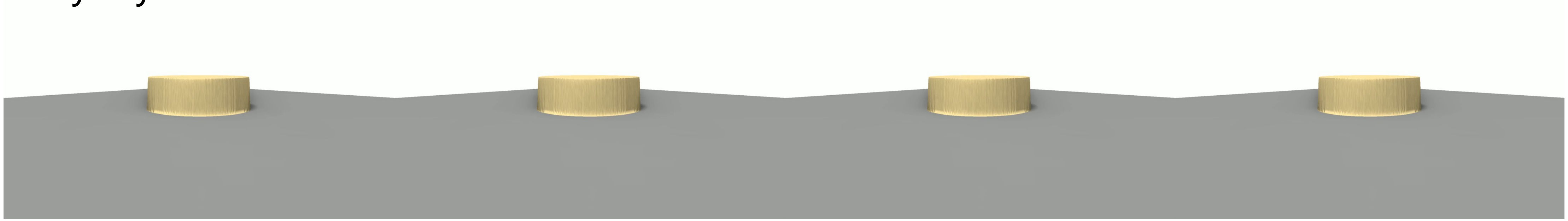
$$c_{\mathrm{c}}(\phi) = \begin{cases} c_0 + \phi\frac{c_1 - c_0}{\phi_1}, & \text{if } \ 0 \leq \phi < \phi_1 \\ c_1 + (\phi - \phi_1)\frac{c_2 - c_1}{\phi_2 - \phi_1}, & \text{if } \phi_1 \leq \phi < \phi_2 \\ c_2\frac{\phi_3 - \phi}{\phi_3 - \phi_2}, & \text{if } \phi_2 \leq \phi \leq \phi_3 \\ 0, & \text{if } \phi > \phi_3. \end{cases}$$

ϕ - Saturation level

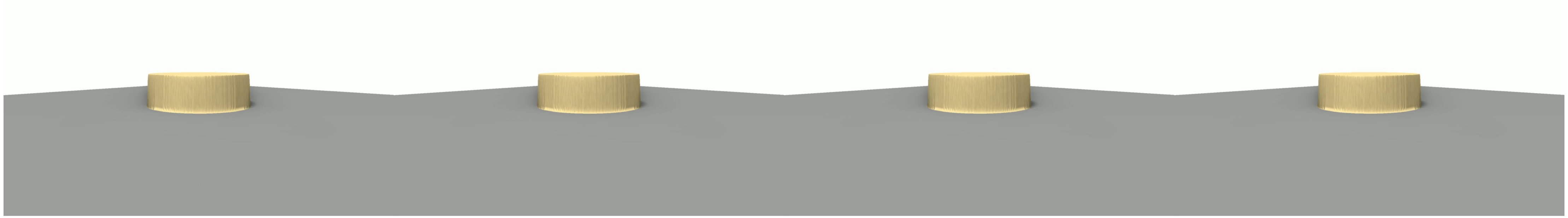


Sand Piling
with different saturation levels

Very dry sand



$\phi = 0.00$ $\phi = 0.03$ $\phi = 0.07$ $\phi = 0.25$

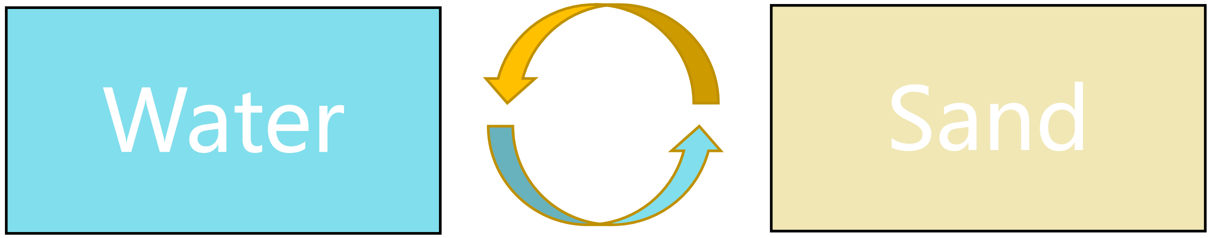


$\phi = 0.30$ $\phi = 0.32$ $\phi = 0.35$ $\phi = 0.40$

Over wet sand

Water-Sand Coupling

Momentum Exchange



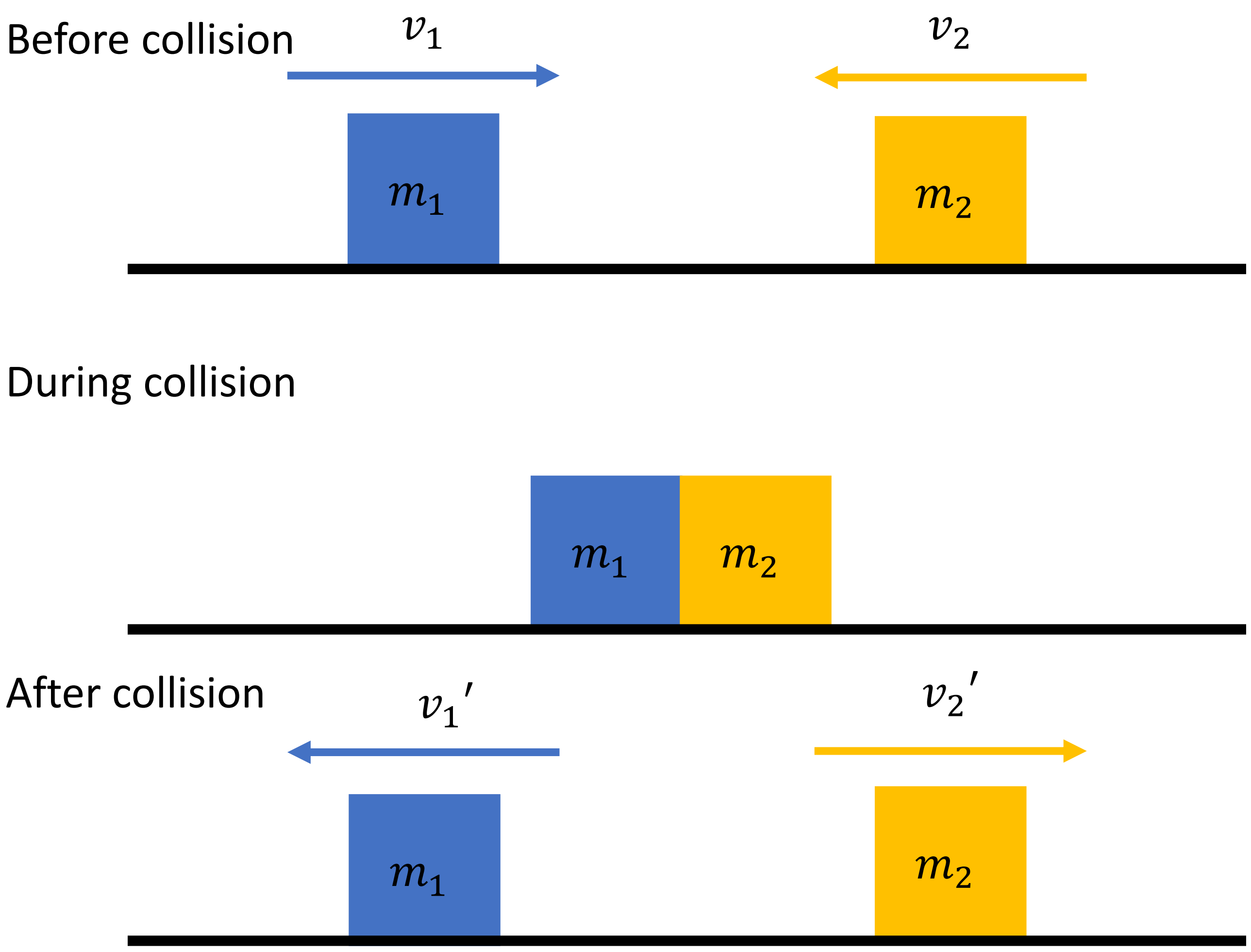
$$\frac{Dh^w}{Dt} = -h^w \nabla \cdot \mathbf{v}^w,$$

$$\frac{D\mathbf{v}^w}{Dt} = -g\nabla(H + h^w) \boxed{+\, \mathbf{a}^w_{\mathcal{M}}}$$

$$\frac{Dh^s}{Dt} = -h^s \nabla \cdot \mathbf{v}^s$$

$$\frac{D\mathbf{v}^s}{Dt} = -g\nabla(H + h^s) + \mathbf{a}^s_{\mathcal{E}} + \mathbf{a}^s_{\mathcal{I}} \boxed{+\, \mathbf{a}^s_{\mathcal{M}}}$$

Momentum Exchange



Momentum conservation

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

Coefficient of restitution

$$\epsilon = \frac{v_2' - v_1'}{v_1 - v_2}$$

$$\rho^w h^w v^w + \rho^s h^s v^s = \rho^w h^w v^{w\prime} + \rho^s h^s v^{s\prime}$$

$$v^{s\prime} - v^{w\prime} = \epsilon(v^w - v^s)$$

Single-layer Shallow Water with Sand

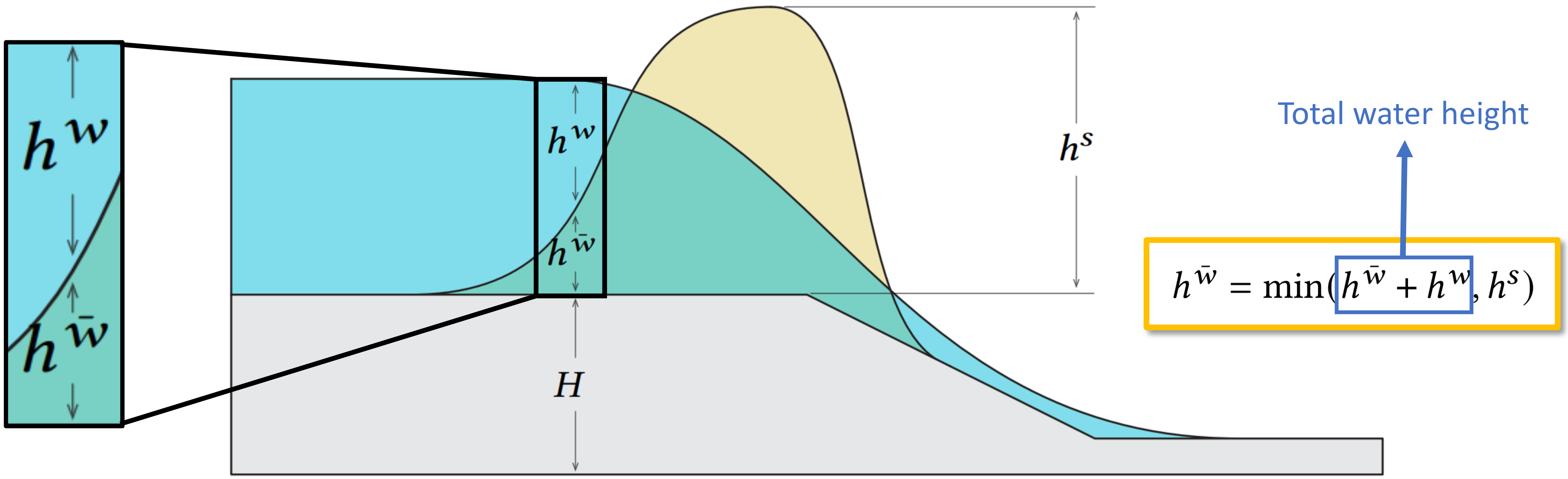
Single-layer Shallow Water with Sand

Water movement on the top is impeded by the underneath sand

Our Water-Sand Simulation Framework

Two-layer SWEs

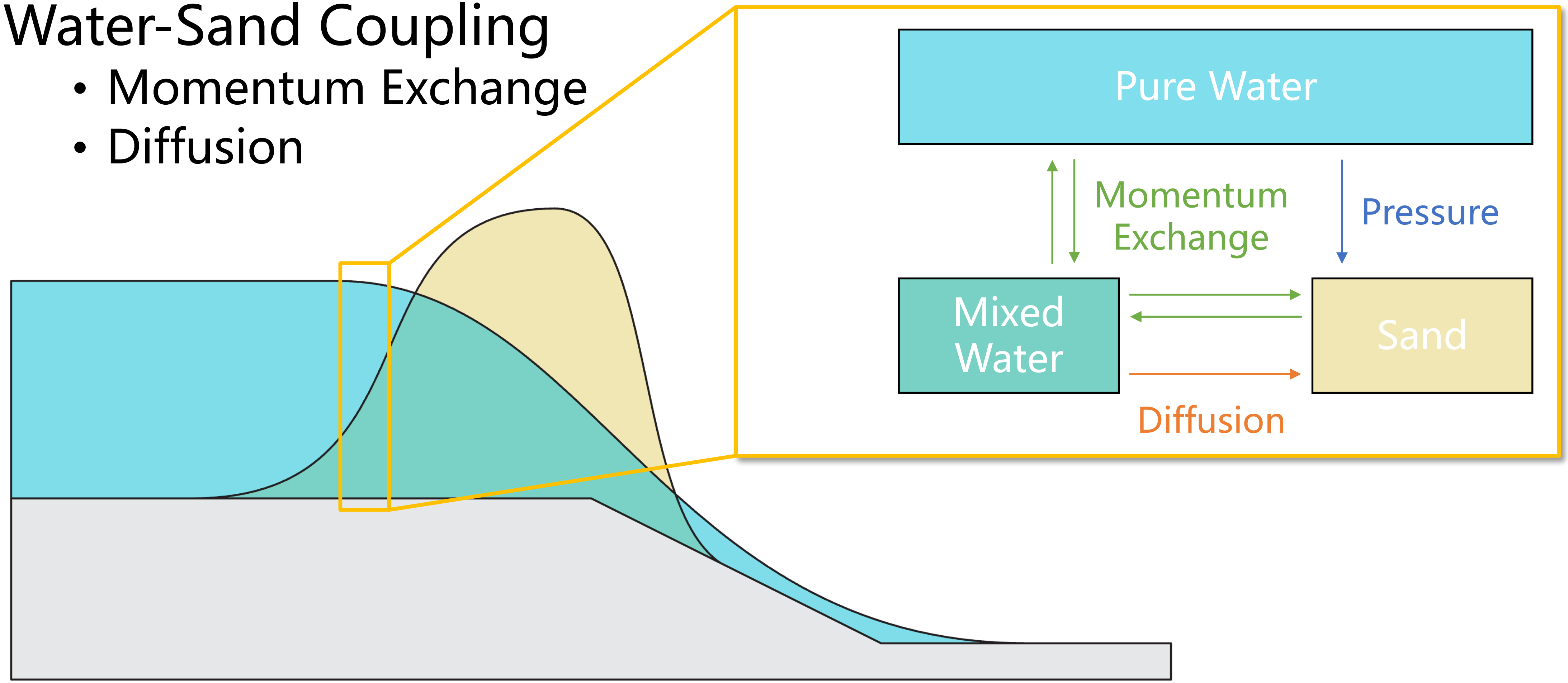
- Pure water h^{w}, the part of water above the sand
- Mixed water $h^{\bar{w}}$, the part of water submerged under sand



Our Water-Sand Simulation Framework

Water-Sand Coupling

- Momentum Exchange
- Diffusion



Water-Sand Coupling

Diffusion

- Conservation of mass

$$\frac{Dh^{\bar{w}}}{Dt} = -h^{\bar{w}}\nabla \cdot \mathbf{v}^{\bar{w}} + \boxed{c_d \nabla \cdot \nabla h^{\bar{w}}}$$

- Conservation of momentum

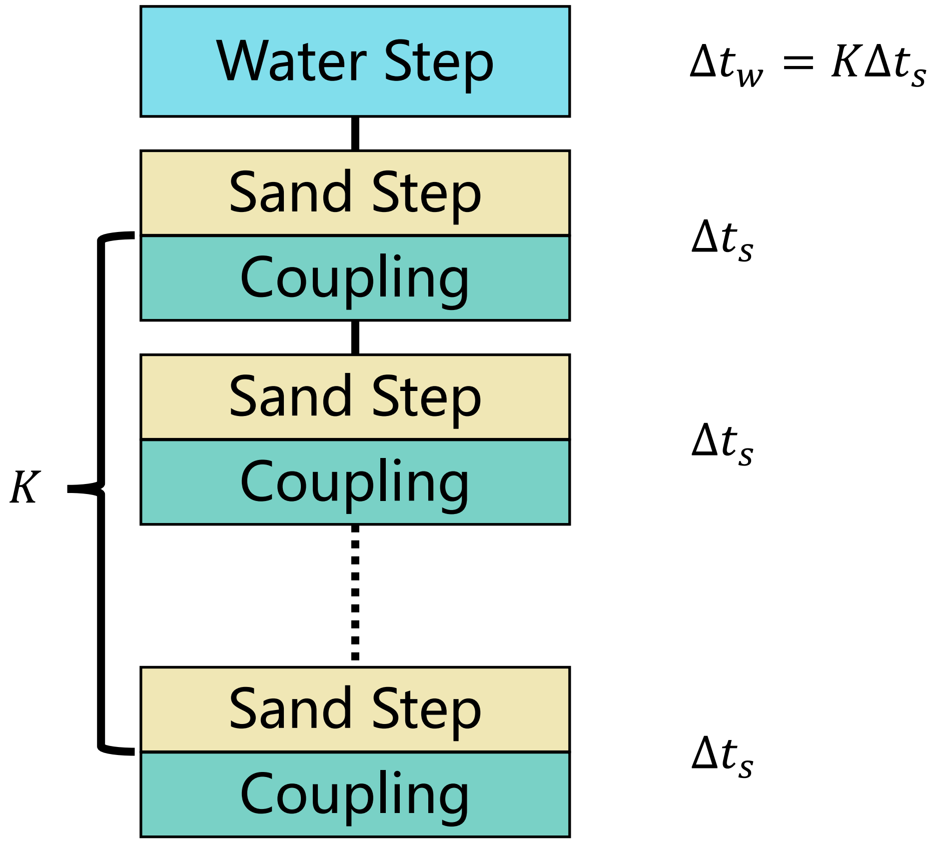
$$\frac{D\mathbf{v}^s}{Dt} = -g\nabla\left(H + h^s + \frac{\rho^w}{\rho^s} h^w\right) + \mathbf{a}^s_{\mathcal{E}} + \mathbf{a}^s_{\mathcal{I}} + \boxed{c_d \left(\nabla \cdot \nabla (h^{\bar{w}} \mathbf{v}^{\bar{w}})\right) / h^{\bar{w}}}$$

Single-layer Shallow Water with Sand

Two-layer Shallow Water with Sand

Implementation Details

- Asynchronous Update



Water Step
- Height/Velocity integration
- Mixed water diffusion

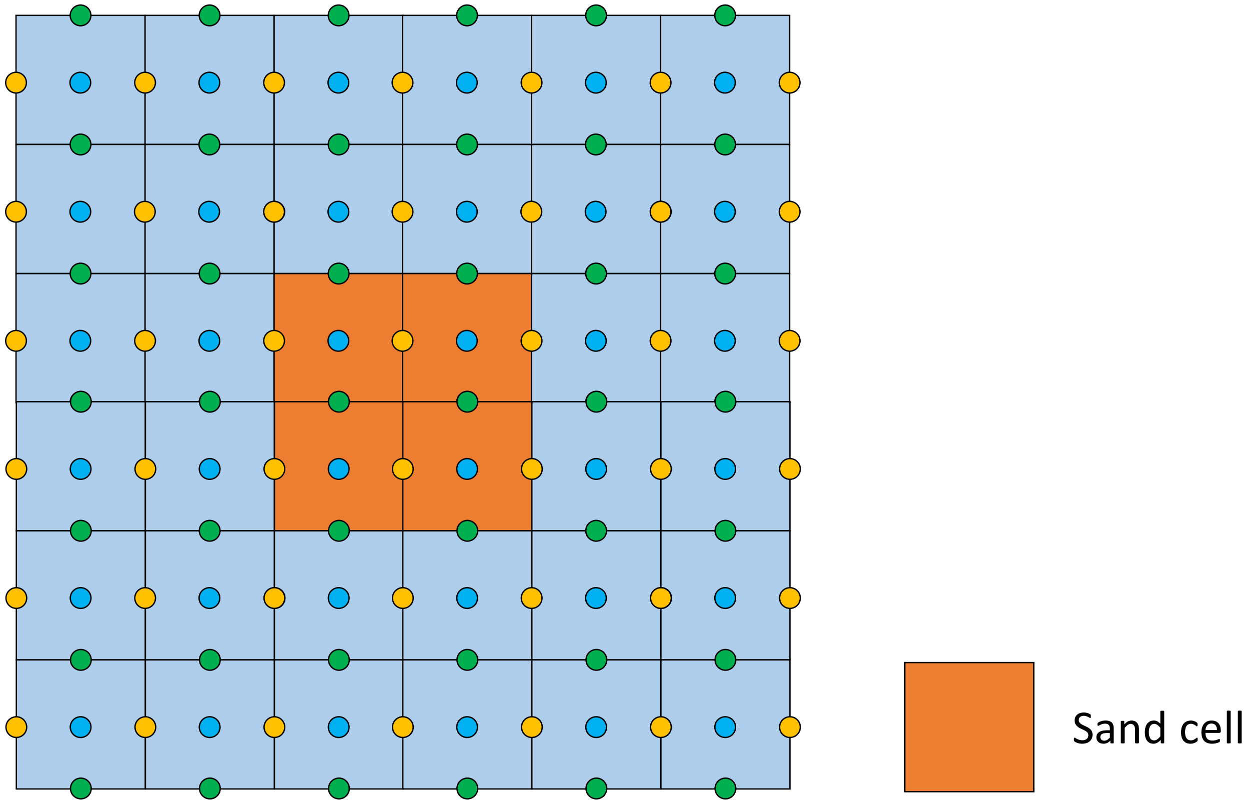
Sand step
- Height/Velocity integration
- Pure water pressure
- Deformation gradient update
- Internal force

Coupling
- Pure/mixed water momentum exchange
- Sand/mixed water momentum exchange

After coupling
- Friction

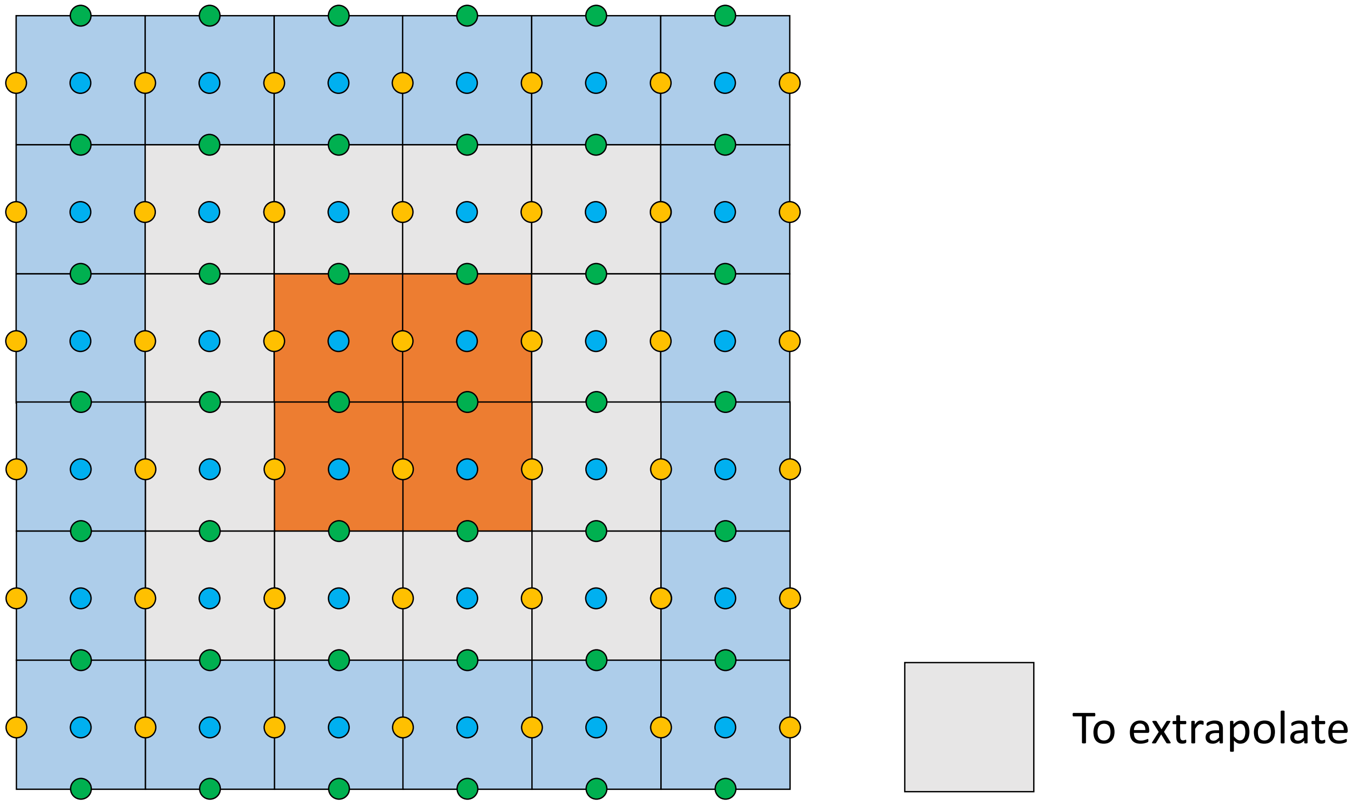
Implementation Details

- Deformation gradient extrapolation



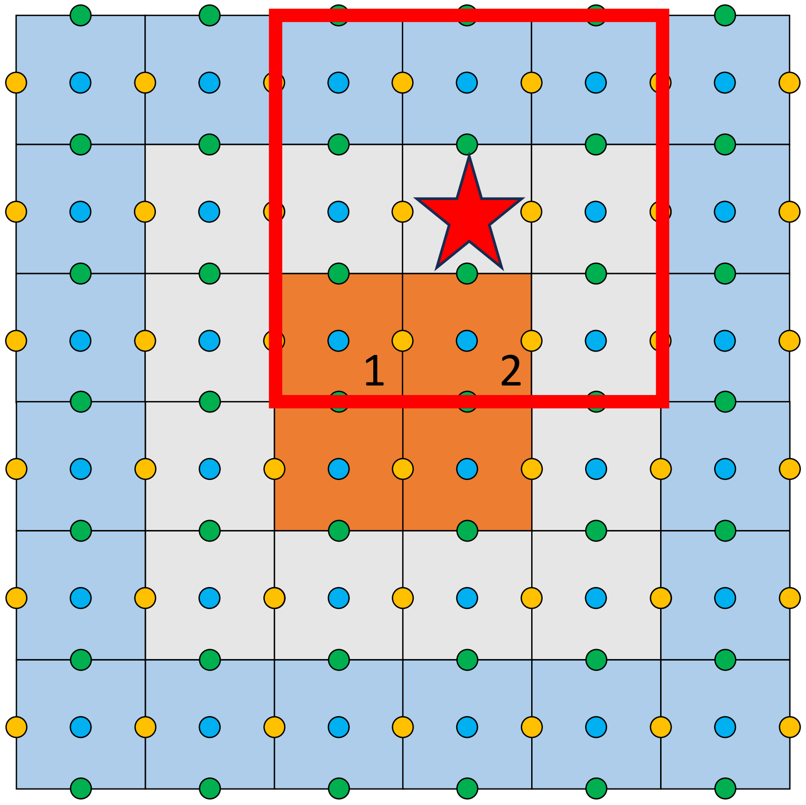
Implementation Details

- Deformation gradient extrapolation



Implementation Details

- Deformation gradient extrapolation



$$F_\star = \frac{F_1 + F_2}{2}$$

Failure case:

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow F_\star = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Implementation Details

- Deformation gradient decomposition

$$\mathrm{F} = \mathrm{R}(\theta)\ (\mathrm{E} + \mathrm{I})$$

- Same case:

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \theta_1 = 0, E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \theta_2 = \pi, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

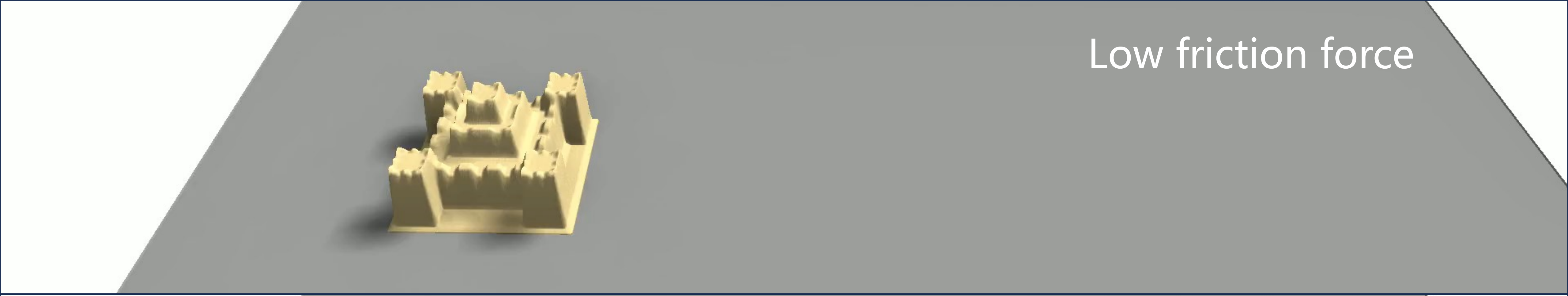
$$\theta_\star = \frac{\theta_1 + \theta_2}{2} = \frac{\pi}{2}, E_\star = \frac{E_1 + E_2}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow F_\star = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

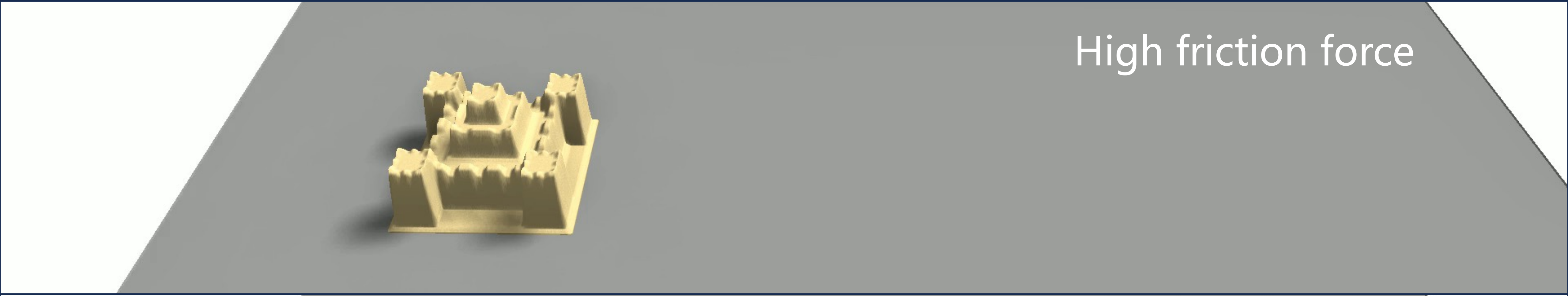
Implementation Details

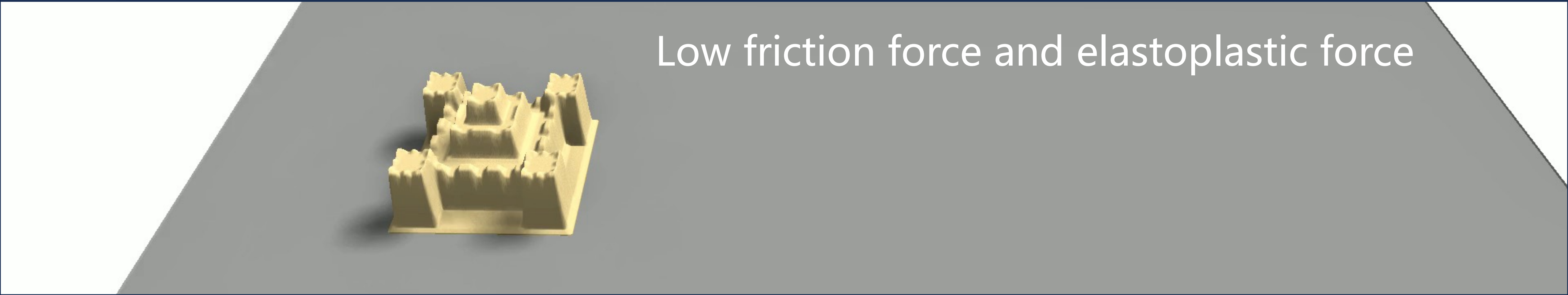
- Conservative advection

$$\frac{D(h^s\theta)}{Dt} = 0 \Rightarrow \theta = \frac{(h^s\theta)}{h^s}$$

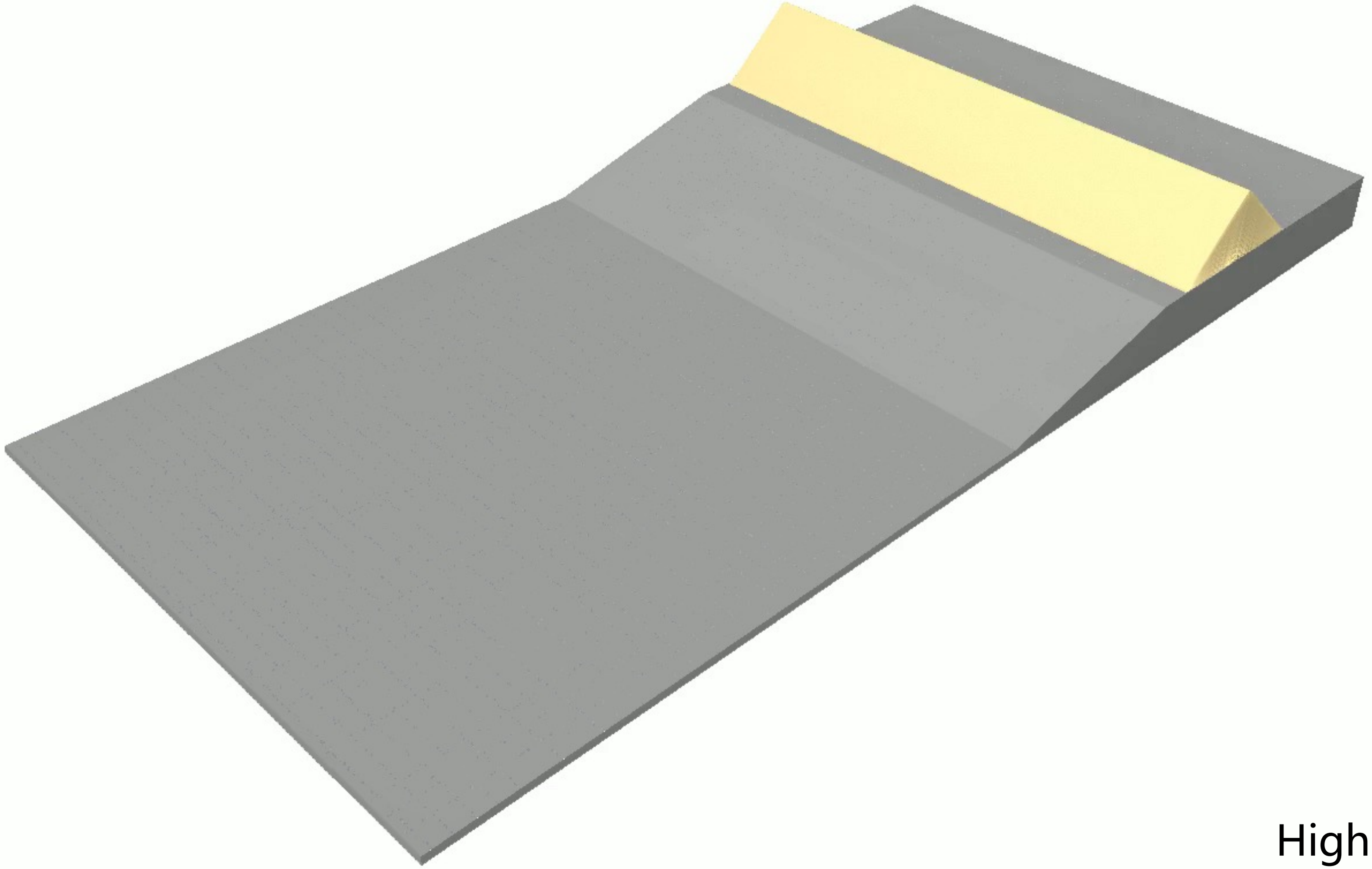
$$\frac{D(h^sE)}{Dt} = 0 \Rightarrow E = \frac{(h^sE)}{h^s}$$


Low friction force


High friction force

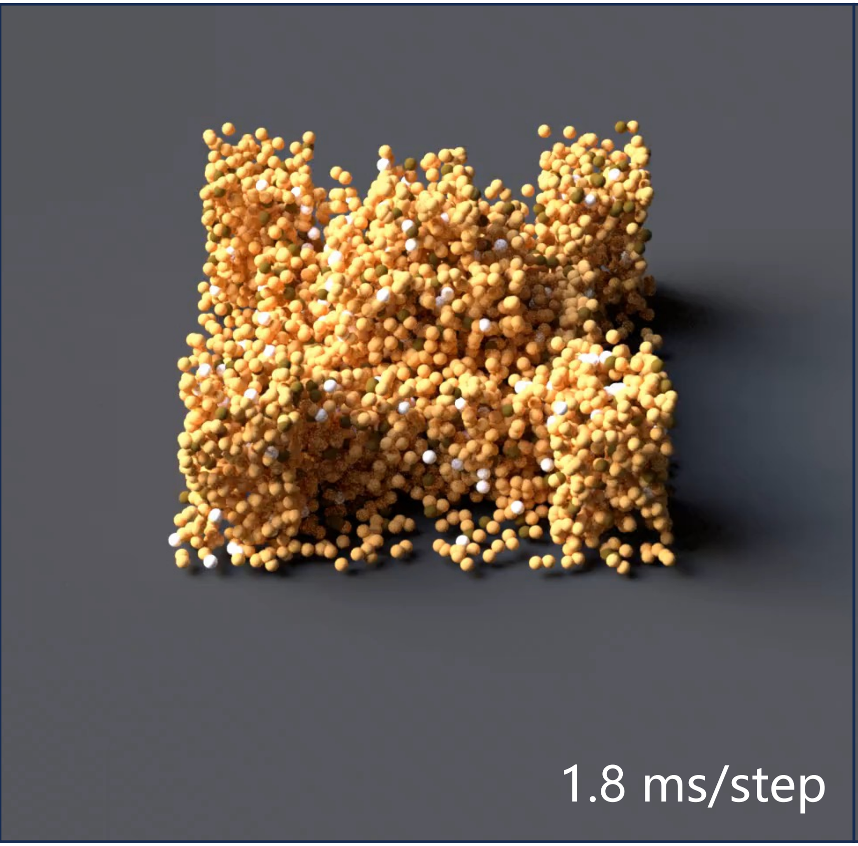

Low friction force and elastoplastic force

Seepage

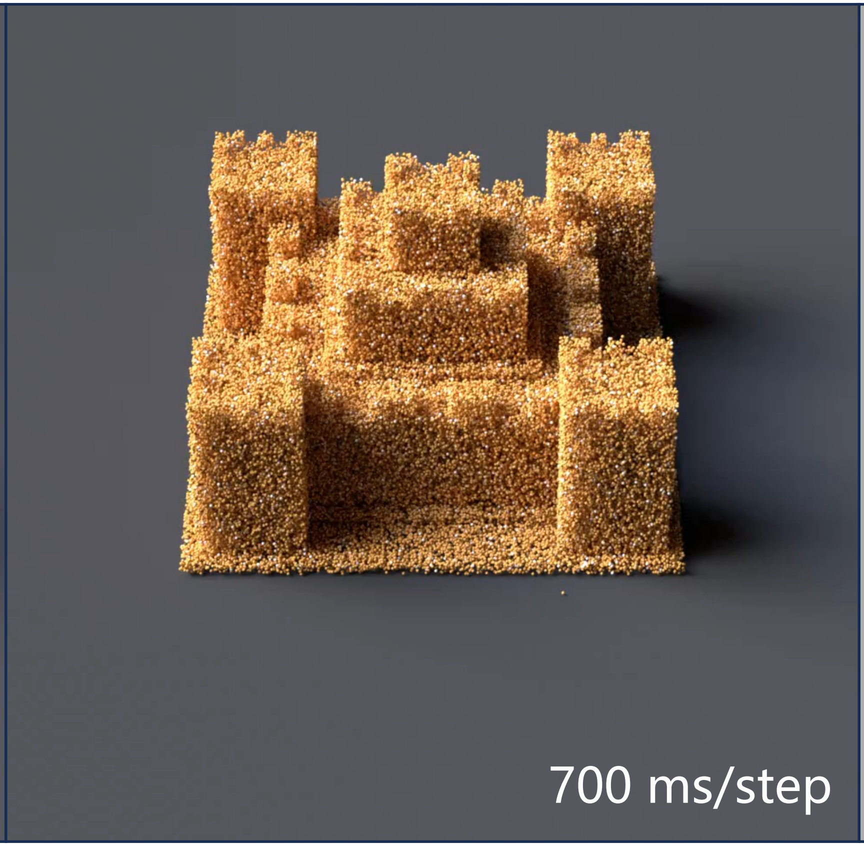


High friction

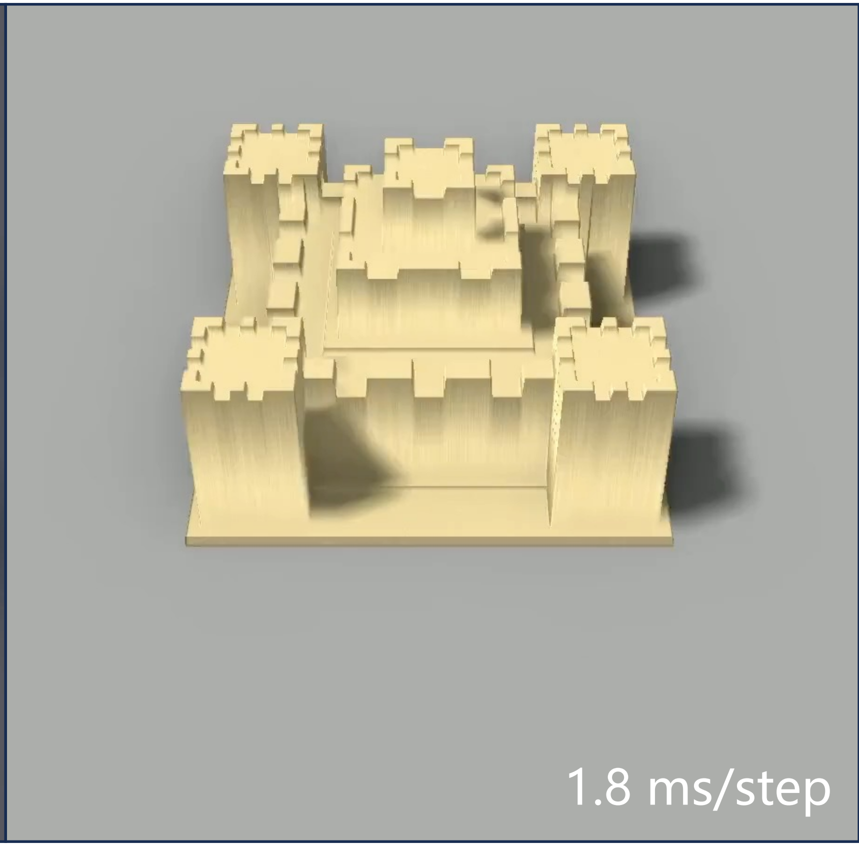
Comparison with MPM



MPM $8K, 64^3$

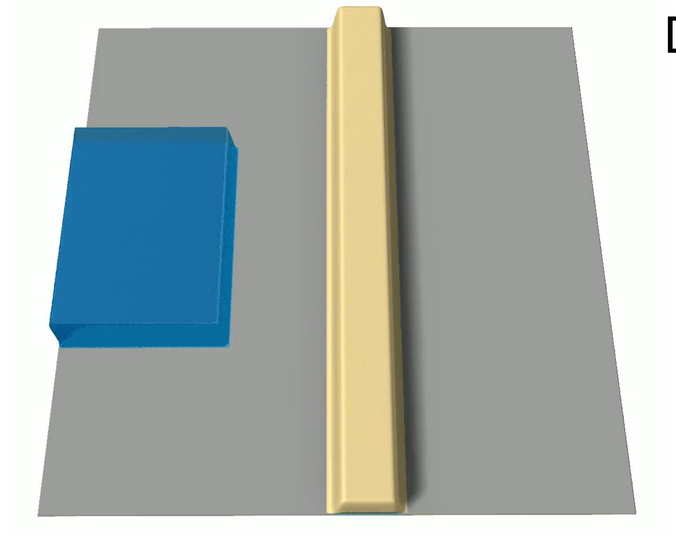


MPM $500K, 256^3$

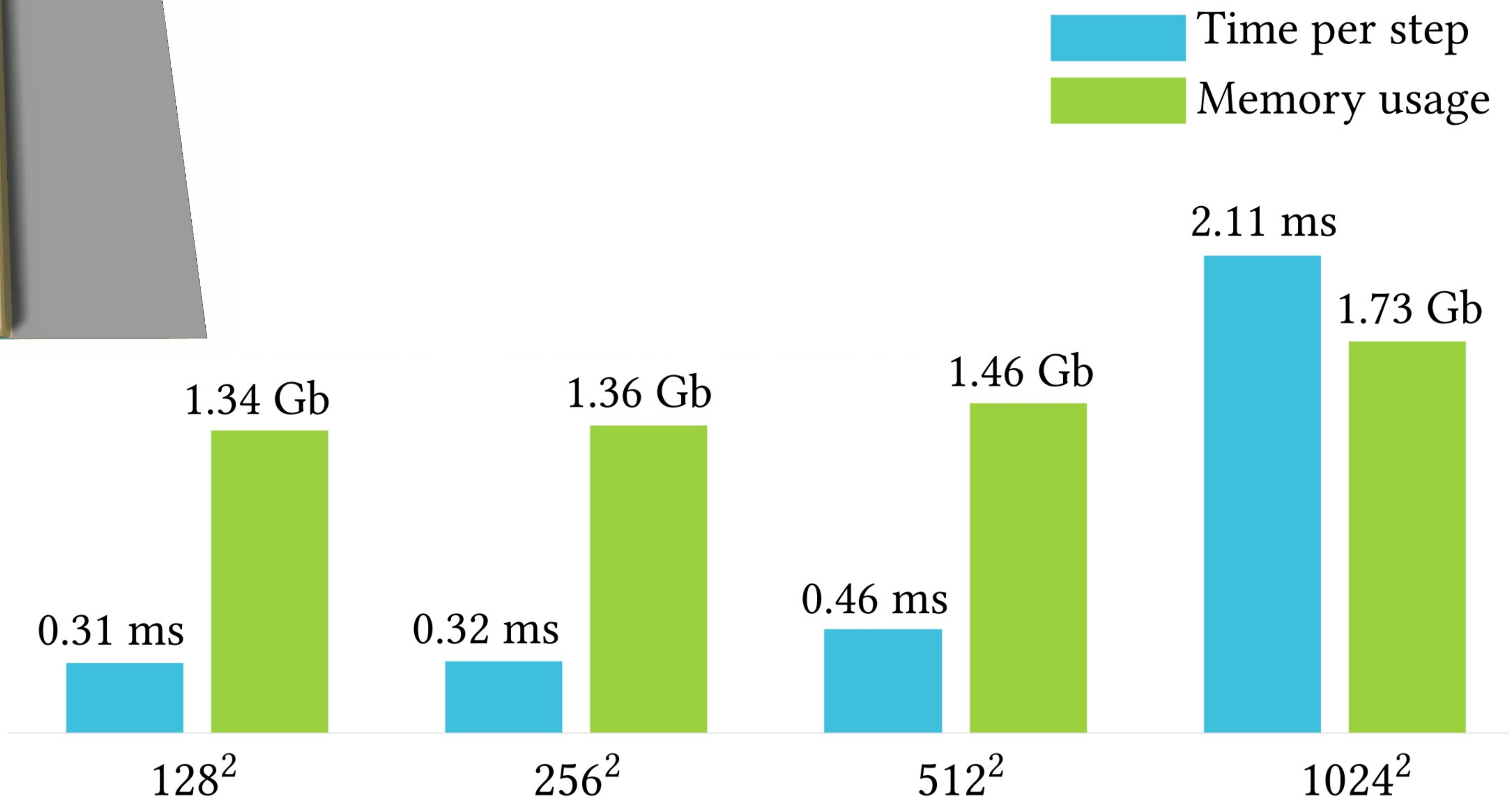


Ours 1024^2

Performance



Dam break



Interactive Demo

Conclusion

A height-field-based real-time framework to simulate sand, water, and their mixture

- A 2.5D governing equation for sand-water mixtures
- A grid-based elastoplastic formulation for sand
- A piecewise linear saturation control function

Q & A

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LIGHTSPEED STUDIOS

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