SIGGRAPH 2022
VANCOUVER+ 8-11 AUG

A FAST UNSMOOTHED AGGREGATION ALGEBRAIC MULTIGRID FRAMEWORK FOR THE LARGE-SCALE SIMULATION OF INCOMPRESSIBLE FLOW

Han Shao, Libo Huang, Dominik L. Michels

CONTENT

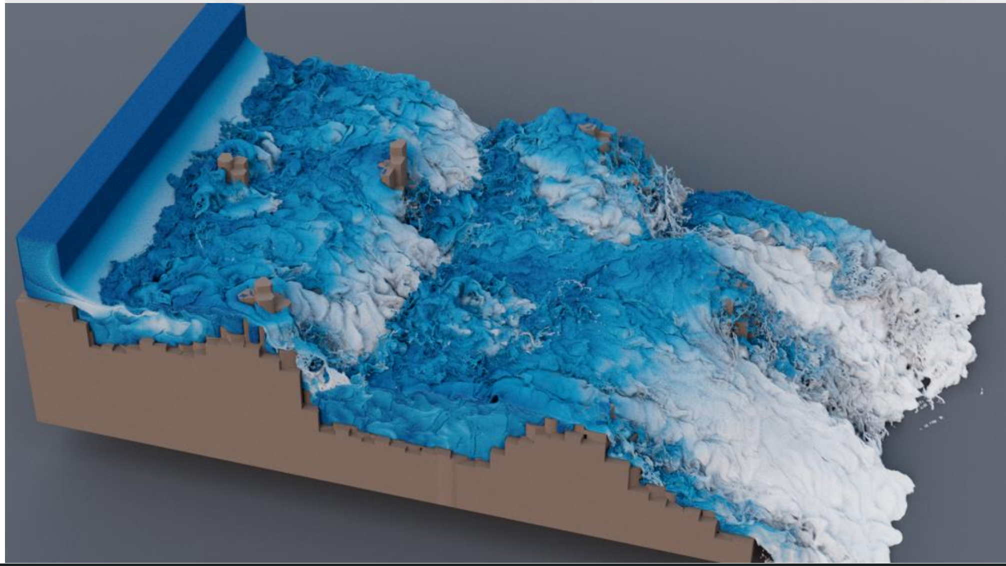
SIGGRAPH 2022
VANCOUVER+ 8-11 AUG

- Background and Motivation
  - Simulation of Incompressible Flow
  - Pressure Poisson Equation
  - Variational Viscosity Equation
- Our framework: UAAMG
  - Multigrid, AMG, Unsmoothed Aggregation
  - Matrix-free Coarsening and Matrix Vector Multiplication
- Experiments and Results
  - Unit Test
  - Complex Scene

- Simulation of Incompressible Flow
  - NS Equation
    - Viscous and non-viscous

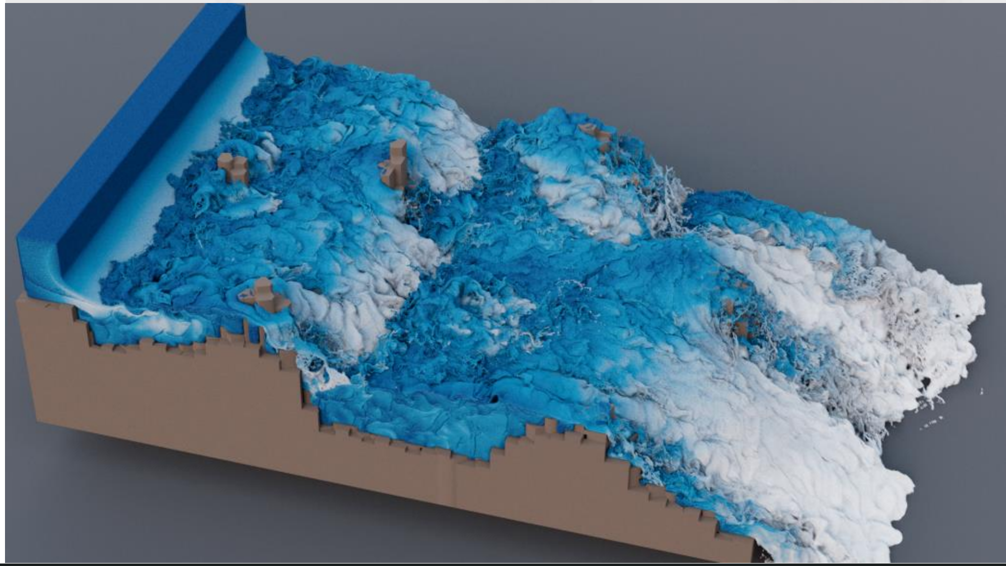
$$\vec{\mathbf{u}}_t + \left(\vec{\mathbf{u}} \cdot \nabla\right)\vec{\mathbf{u}} = -\frac{\nabla p}{\rho} + \frac{\nabla \cdot \tau}{\rho} + \vec{\mathbf{g}}$$

$$\tau = \mu\left(\nabla \vec{\mathbf{u}} + \nabla \vec{\mathbf{u}}^T\right)$$



BACKGROUND AND MOTIVATION

- Simulation of Incompressible Flow
  - Projection Method (Chorin, 1967)
  - Variational Viscosity (Batty, 2008)
    - Advection: $\frac{\vec{\mathbf{u}}^{*} - \vec{\mathbf{u}}^{n}}{\Delta t} + \left(\vec{\mathbf{u}}^{n} \cdot \nabla\right)\vec{\mathbf{u}}^{n} = \vec{\mathbf{g}}^{n}$
    - Pressure: $-\nabla^2 p = \frac{\rho}{\Delta t}\nabla \cdot \vec{\mathbf{u}}^{*}, \quad \frac{\vec{\mathbf{u}}^{**} - \vec{\mathbf{u}}^{*}}{\Delta t} = -\frac{\nabla p}{\rho}$
    - Viscosity: $\frac{\vec{\mathbf{u}}^{***} - \vec{\mathbf{u}}^{**}}{\Delta t} = \frac{\nabla \cdot \tau}{\rho}$



SIGGRAPH 2022 VANCOUVER+ 8-11 AUG

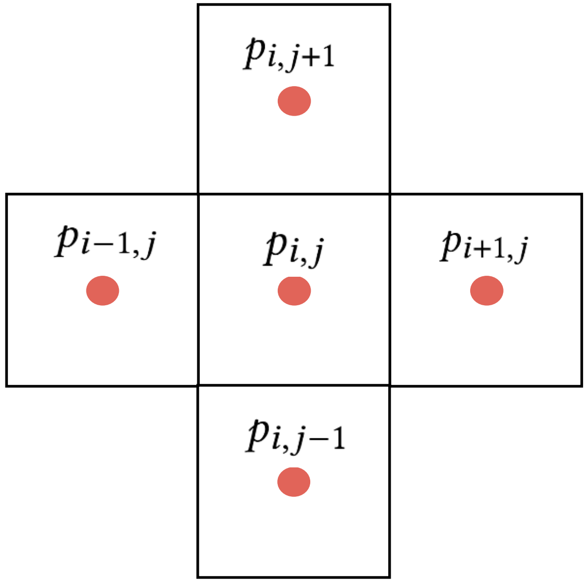
- Pressure Poisson Equation
  - Discretization

$$-\nabla^2 p = \frac{\rho}{\Delta t} \nabla \cdot \vec{\mathbf{u}}^*$$

$$C_1 p_{i-1,j} + C_2 p_{i+1,j} + C_3 p_{i,j-1} + C_4 p_{i,j+1} + C_5 p_{i,j} = C_0$$

Parallelly Calculating the Matrix-Vector Multiplication in a Matrix-Free Fashion!

2D Cartesian Grid



BACKGROUND AND MOTIVATION

- Pressure Poisson Equation
  - SIMD - Single Instruction/Multiple Data

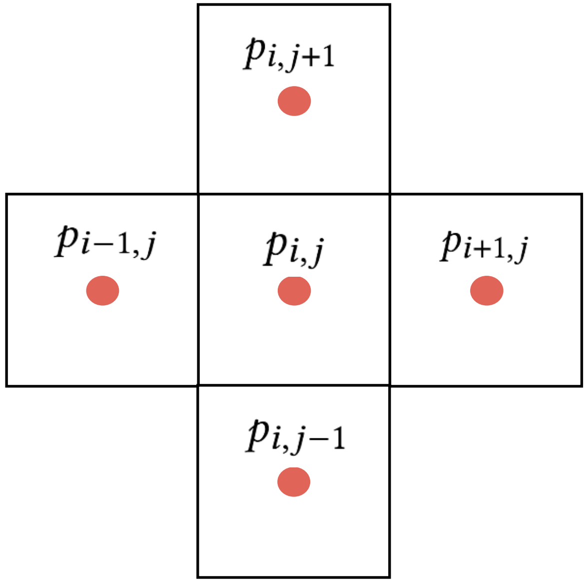
$$C_1 p_{i-1,j} + C_2 p_{i+1,j} + C_3 p_{i,j-1} + C_4 p_{i,j+1} + C_5 p_{i,j} = C_0$$

A floating point number 32 bits

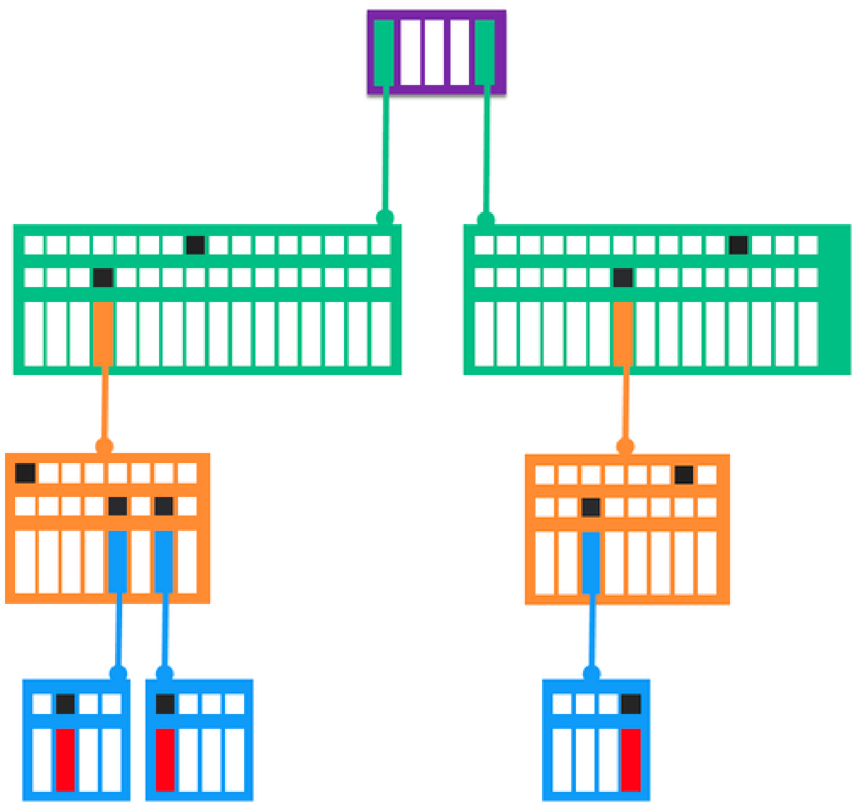
Intel AVX register 256 bits

Parallel computation of the above operation from j=k to j=k+8

2D Cartesian Grid



- Pressure Poisson Equation
  - OpenVDB
    - Variant of B+Tree
    - A leaf Node contains 8X8X8 voxels
    - Improve cache locality

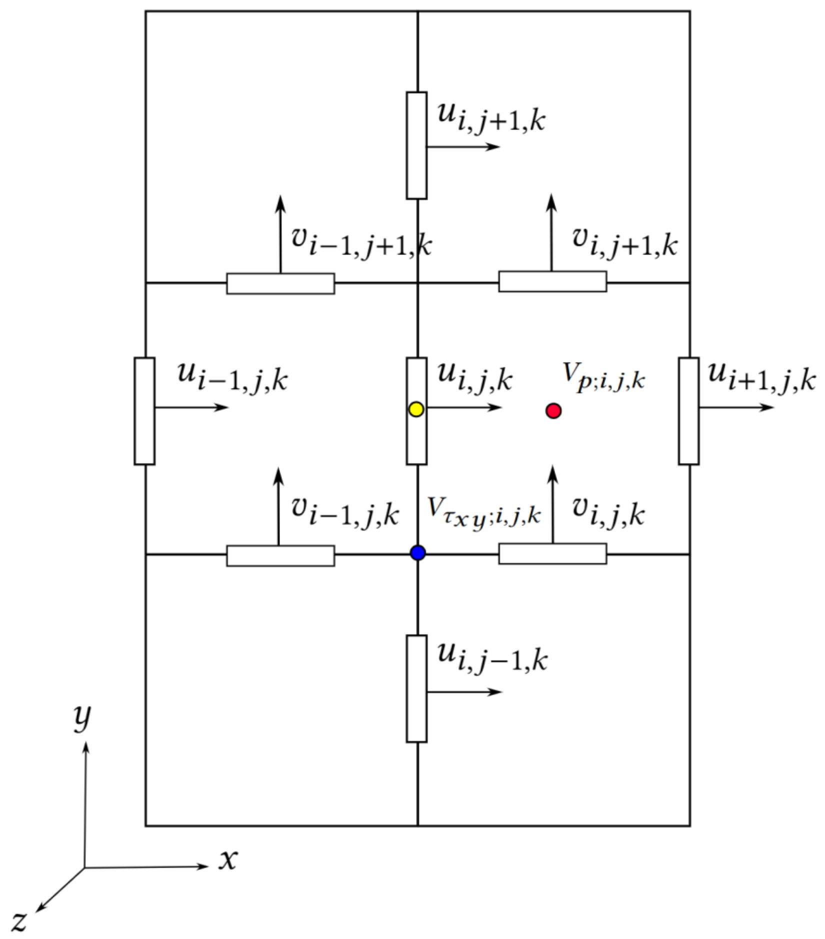


BACKGROUND AND MOTIVATION

- Pressure Poisson Equation
  - Linear Solver
    - Gauss-Seidel, Damped/Scheduled Jacobian, SOR
    - DPCG/ICPCG
    - Geometric multigrid method
    - Adaptive octree approach
    - UAMMG
    - How to be compatible with SIMD implementation?

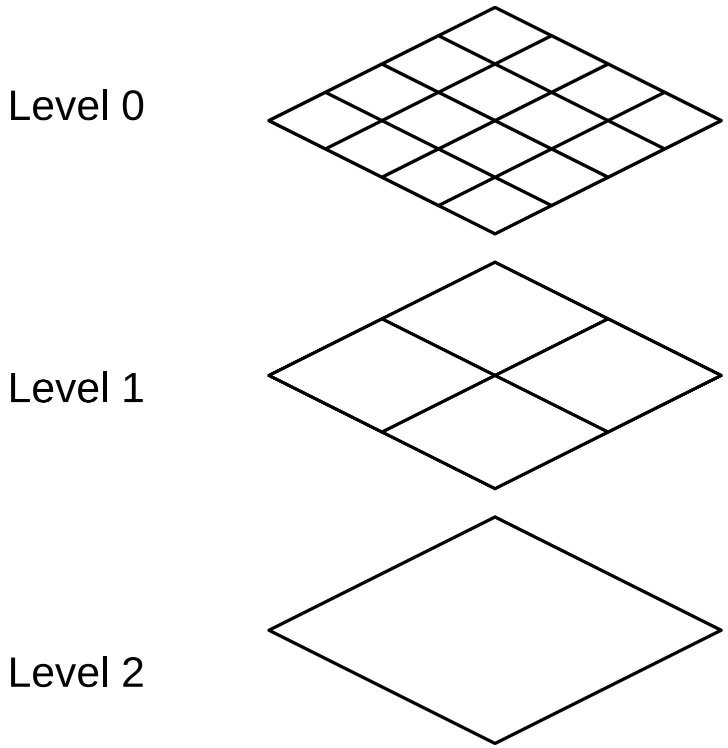
BACKGROUND AND MOTIVATION

- Variational Viscosity Equation
  - Discretization
    - Poisson Equation + Cross Term
    - Three channels of velocity
  - Linear Solver
    - Inefficient smoother for multigrid



UAAMG

- Multigrid and AMG
  - High-frequency error can be quickly smoothed
  - Transform ‘low-frequency' error into “high-frequency”
    - Solve on fine grid
    - Compute residual of the equation
    - Restriction, down-sample residual to coarse grid
    - Solve on coarse grid
    - Prolongation, interpolate correction to fine grid
    - Update the fine grid solution



UAAMG

- Multigrid and AMG

Fine level, equation: $A_h u_*^h = f^h$, error: $e_{\text{old}}^h = u_*^h - u_{\text{old}}^h$

Residual: $A_h e_{\text{old}}^h = r_{\text{old}}^h = f^h - A_h u_{\text{old}}^h$

Restriction to coarse level: $r^H = R r_{\text{old}}^h$

Coarse level, equation: $A_H e^H = r^H$

Prolongation and correction: $u_{\text{new}}^h = u_{\text{old}}^h + \alpha P e^H$

- Coarsening: get matrix A_H
- Geometric Multigrid:
  - Coarse grid matrix depends on the discretization
- Algebraic Multigrid:
  - Coarse grid matrix is calculated following Galerkin Principle
    $A_H := R A_h P$

UAAMG

- Multigrid and AMG
  - Algorithm Overview
  - Key Components:
    - Smoother
    - Matrix on each level A
    - Prolongation P, Restriction R
    - Matrix Vector Multiplication

Algorithm 1: Recursive μ-cycle.

Input: initial guess u_l, current level l, right hand side f_l
Output: updated u_l
if $l = nLevels - 1$ **then**
  solve $A_l u_l = f_l$ directly;
  return u_l;
end
apply ν times relaxations to $A_l u_l = f_l$; `/* pre-smooth */`
$r_l = f_l - A_l u_l$; `/* calculate this level residual */`
$r_{l+1} = R_l r_l$; `/* restriction */`
$e_{l+1} \leftarrow 0$;
apply μ times $e_{l+1} \leftarrow \mu$-Cycle$(e_{l+1}, l+1, r_{l+1})$;
if *Poisson and Preconditioner* **then**
  $u_l \leftarrow u_l + 2P_l e_{l+1}$; `/* prolongation */`
else
  $u_l \leftarrow u_l + P_l e_{l+1}$; `/* prolongation */`
end
apply ν times relaxations to $A_l u_l = f_l$; `/* post-smooth */`
return u_l;

UAAMG

- Unsmoothed Aggregation
  - Aggregation: Each fine mesh element maps to only one coarse mesh element
  - Unsmooth: Piece-wise constant restriction/prolongation
  - Geometric information: 1 coarse voxel contains 8 fine voxels

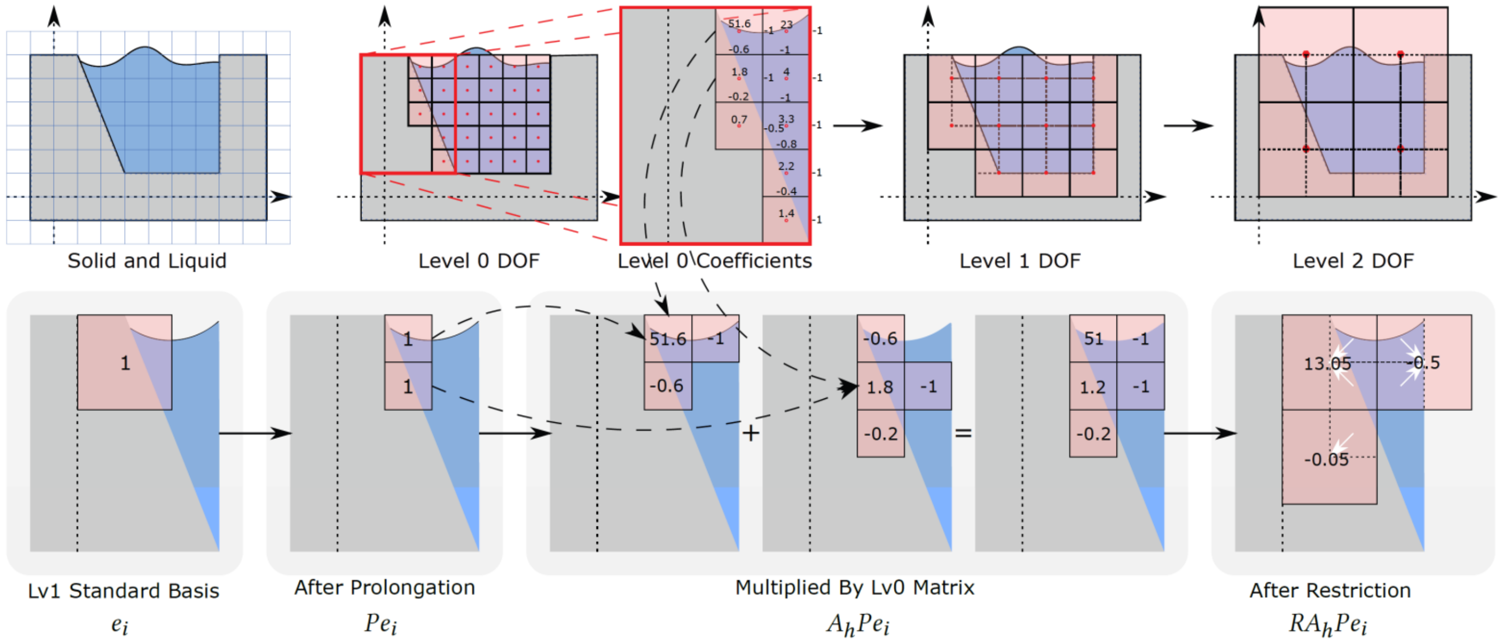
$$R_{ij} = \begin{cases} 1/8, & \text{if coarse voxel } i \text{ covers fine voxel } j, \\ 0, & \text{otherwise.} \end{cases}$$

$$P = 8R^T$$

Galerkin Principle: $A_H := RA_hP$

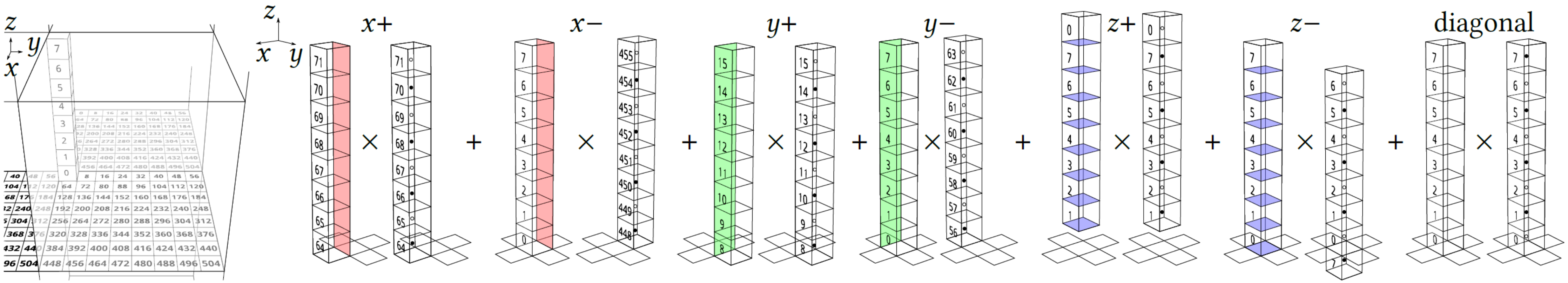
UAAMG

- Matrix-free Coarsening
  - Explicit matrix multiplication for $A_H := RA_hP$ is avoided
  - Using geometric information
    - Diagonal:
    - Off-diagonal:



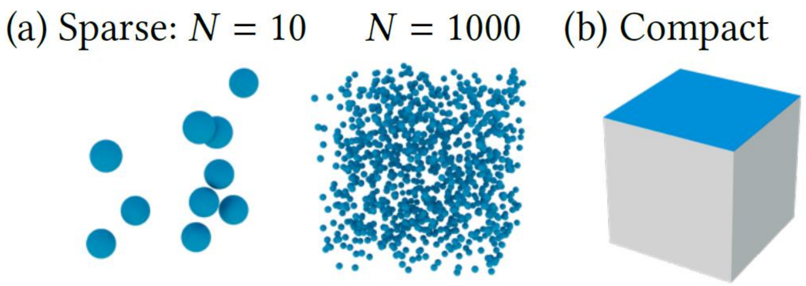
SIGGRAPH 2022 VANCOUVER+ 8-11 AUG

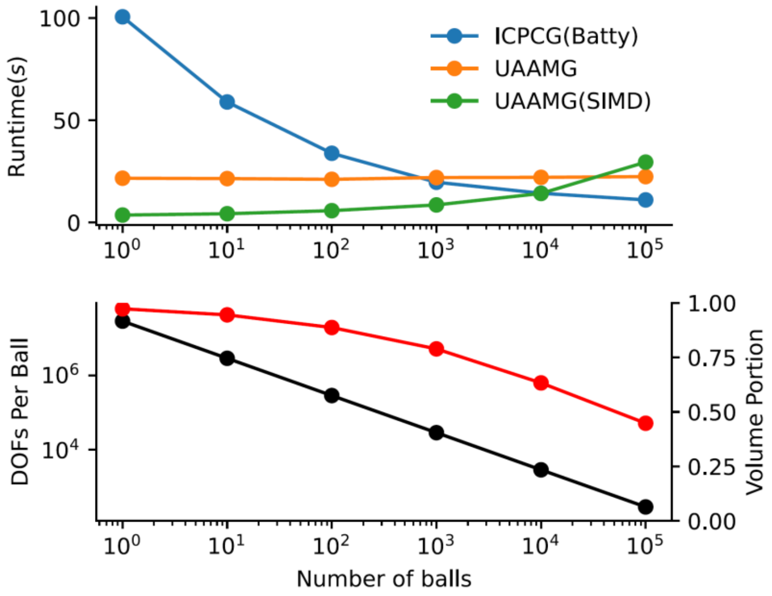
- Matrix-free Matrix Vector Multiplication



- Matrix-free matrix vector multiplication is done by:
  - Multiplying coefficient SIMD vector and DOF SIMD vector
  - Summing up
- Red: x direction
- Green: y direction
- Blue: z direction

- Unit Test – Pressure Poisson Equation
  - Compact Scene
    - Compared to best baseline ICPCG
    - 20X, 59X, 147X for different resolutions
      - #DOFs, 2M, 17M, 135M
  - Sparse Scene
    - SIMD perform well unless in extreme sparse scene



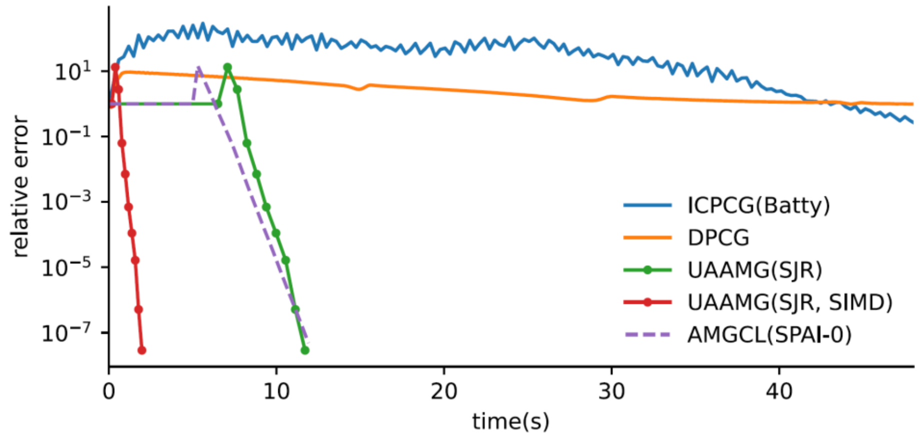


EXPERIMENTS AND RESULTS

SIGGRAPH 2022 VANCOUVER+ 8-11 AUG

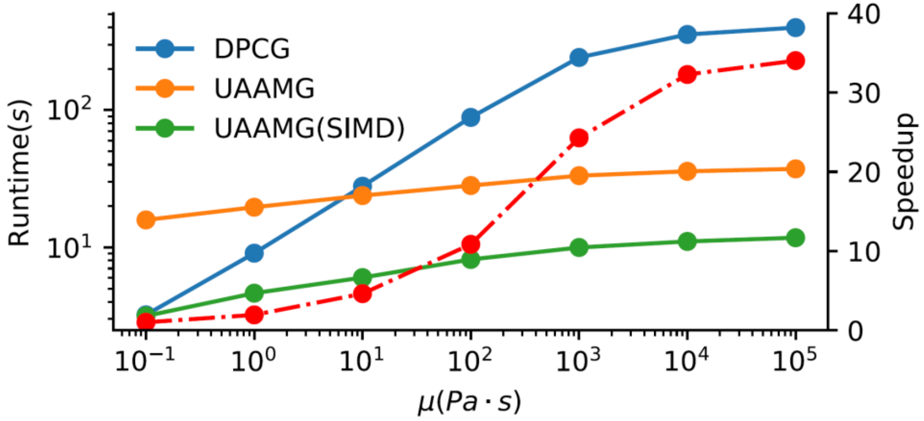
- Unit Test – Pressure Poisson Equation
  - SMID vs Naïve Approach
  - Matrix-free Coarsening
  - Default Tree Trimming
  - Compared to AMGCL

Resolution 256



Resolution		128	256	512	1, 024
# DOFs		2, 146, 685	16, 974, 589	135, 005, 693	1, 076, 890, 621
DPCG		1,323 iters 18.6s	4,544 iters 584.7s	9,322 iters 8,756.4s	-
ICPCG (Eigen)		782 iters 108.9s	2,538 iters 6,407.1s	-	-
ICPCG (Batty)		195 iters 7.5s	395 iters 116.8s	903 iters 1,943.7s	-
UAAMG	DJ	11 iters 691ms (746ms)	11 iters 6.3s (6.4s)	11 iters 50.3s (53.2s)	-
	SRJ	9 iters 569ms (747ms)	9 iters 5.2s (6.5s)	9 iters 41.5s (53.7s)	-
	RBGS	7 iters 595ms (766ms)	7 iters 5.8s (6.5s)	8 iters 53.1s (54.6s)	-
	SPAI-0	12 iters 754ms (732ms)	12 iters 7.0s (7.0s)	13 iters 60.3s (54.7s)	-
UAAMG (SIMD)	DJ	11 iters 371ms (61ms)	11 iters 2,161ms (174ms)	11 iters 14.9s (935ms)	11 iters 111.6s (6.6s)
	SRJ	9 iters **308ms (60ms)**	9 iters **1,789ms (180ms)**	9 iters **12.3s (956ms)**	10 iters **102.9s (6.8s)**
	SRJ, NT	9 iters 376ms (60ms)	9 iters 2,534ms (198ms)	9 iters 18.7s (1,178ms)	10 iters 162.5s (8.9s)
	RBGS	7 iters 332ms (62ms)	7 iters 1,896ms (173ms)	8 iters 14.6s (915ms)	8 iters 108.7s (6.7s)
	SPAI-0	12 iters 424ms (58ms)	12 iters 2,457ms (181ms)	13 iters 18.2s (918ms)	13 iters 137.4s (6.7s)
AMGCL	DJ	18 iters 634ms (702ms)	24 iters 7.1s (4.8s)	26 iters 71.3s (40.6s)	-
	SPAI-0	17 iters 600ms (703ms)	23 iters 6.9s (5.0s)	25 iters 64.0s (38.8s)	-

- Unit Test – Variational Viscosity Equation
  - Different Resolutions
    - Compared to best baseline DPCG for low/high viscosity coefficient
    - 0.62X, 1.98X, 3.09X, 3.99X for low viscosity
    - 8.92X, 32.35X, 51.22X, 63.74X for high viscosity
      - #DOFs, 2M, 15M, 49M
  - Speedup vs. Viscosity Coefficient
    - UAAMG performs well for stiff cases

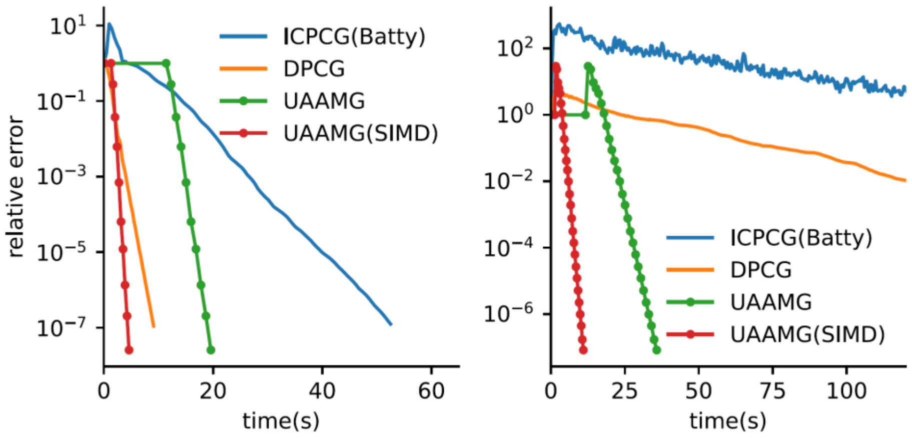


EXPERIMENTS AND RESULTS

SIGGRAPH 2022 VANCOUVER+ 8-11 AUG

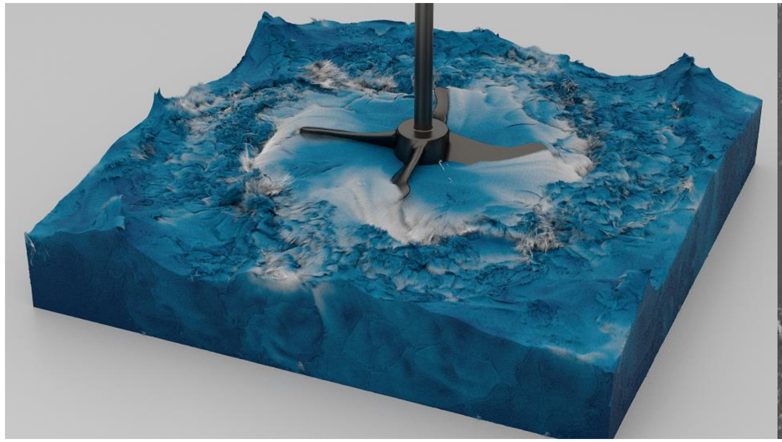
- Unit Test – Variational Viscosity Equation
  - SMID vs Naïve Approach
  - Matrix-free Coarsening
  - Default Tree Trimming
  - High/Low viscosity

Resolution 256



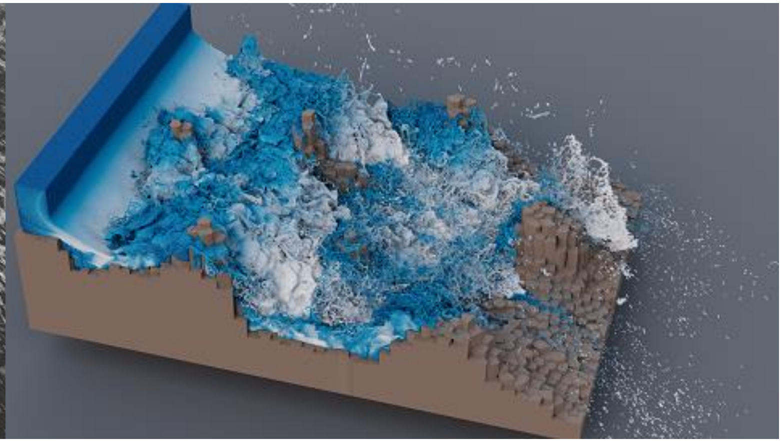
Resolution		128	256	384	512
# DOFs		1, 892, 865	14, 627, 457	48, 805, 069	115, 018, 077
$\mu(1\ \mathrm{Pa}\cdot\mathrm{s})$	DPCG	33 iters **540ms**	65 iters 9.1s	98 iters 43.9s	133 iters 133.5s
	ICPCG (Batty)	19 iters 1,981ms	108 iters 52.5s	138 iters 216.4s	164 iters 572.5s
	UAAMG	6 iters 640ms (1,513ms)	9 iters 8.2s (11.4s)	10 iters 32.1s (40.4s)	11 iters 82.6s (100.8s)
	UAAMG (SIMD)	6 iters 417ms (454ms)	9 iters **3.3s (1.3s)**	10 iters **10.8s (3.4s)**	11 iters **26.1s (7.4s)**
	UAAMG (SIMD, NT)	6 iters 510ms (448ms)	9 iters 4.8s (1.2s)	10 iters 16.6s (3.0s)	11 iters 42.1s (6.6s)
$\mu(10^4\ \mathrm{Pa}\cdot\mathrm{s})$	DPCG	1,155 iters 18.3s	2,705 iters 355.8s	4,706 iters 1,798.2s	5,375 iters 5,252.2s
	ICPCG (Batty)	484 iters 26.7s	1,186 iters 495.2s	2,154 iters 2,951.3s	-
	UAAMG	23 iters 2.3s (1.8s)	27 iters 24.1s (11.7s)	30 iters 94.8s (50.6s)	32 iters 242.1s (98.9s)
	UAAMG (SIMD)	23 iters **1.6s (451ms)**	27 iters **9.7s (1.3s)**	30 iters **31.8s (3.3s)**	32 iters **74.9s (7.5s)**
	UAAMG (SIMD, NT)	23 iters 1.9s (439ms)	27 iters 13.7s (1.2s)	30 iters 47.7s (3.0s)	32 iters 118.5s (6.7)

- Complex Scene - non-viscous flow



2.94X

5.32X

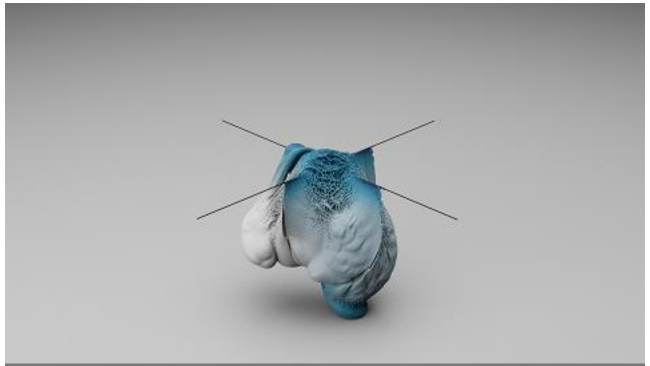


3.55X

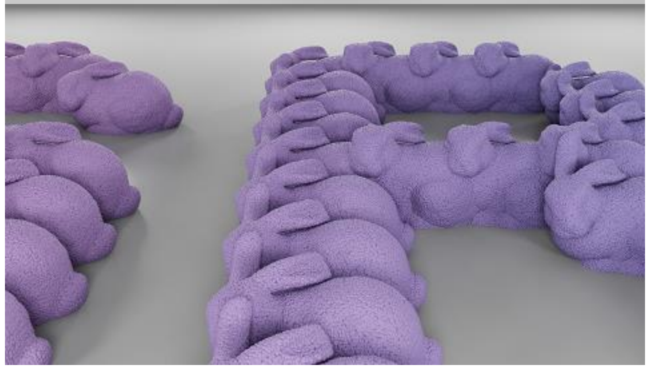
- Compared to Houdini, adaptive octree approach, per-step statistics

EXPERIMENTS AND RESULTS

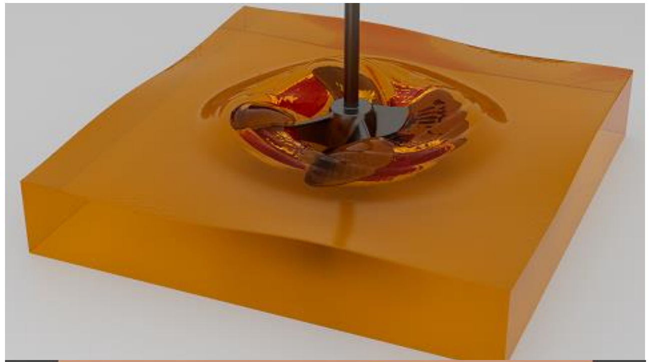
- Complex Scene - viscous flow



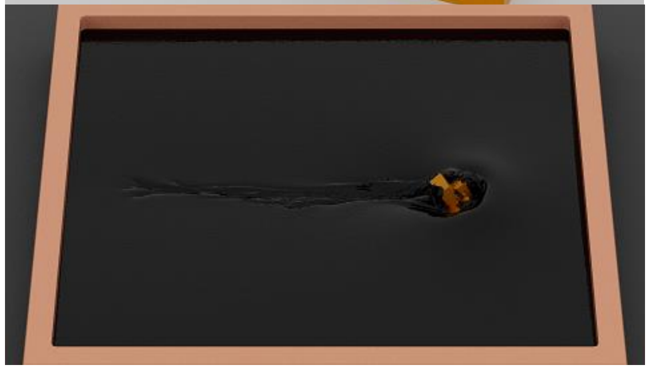
6.73X



4.20X



2.00X



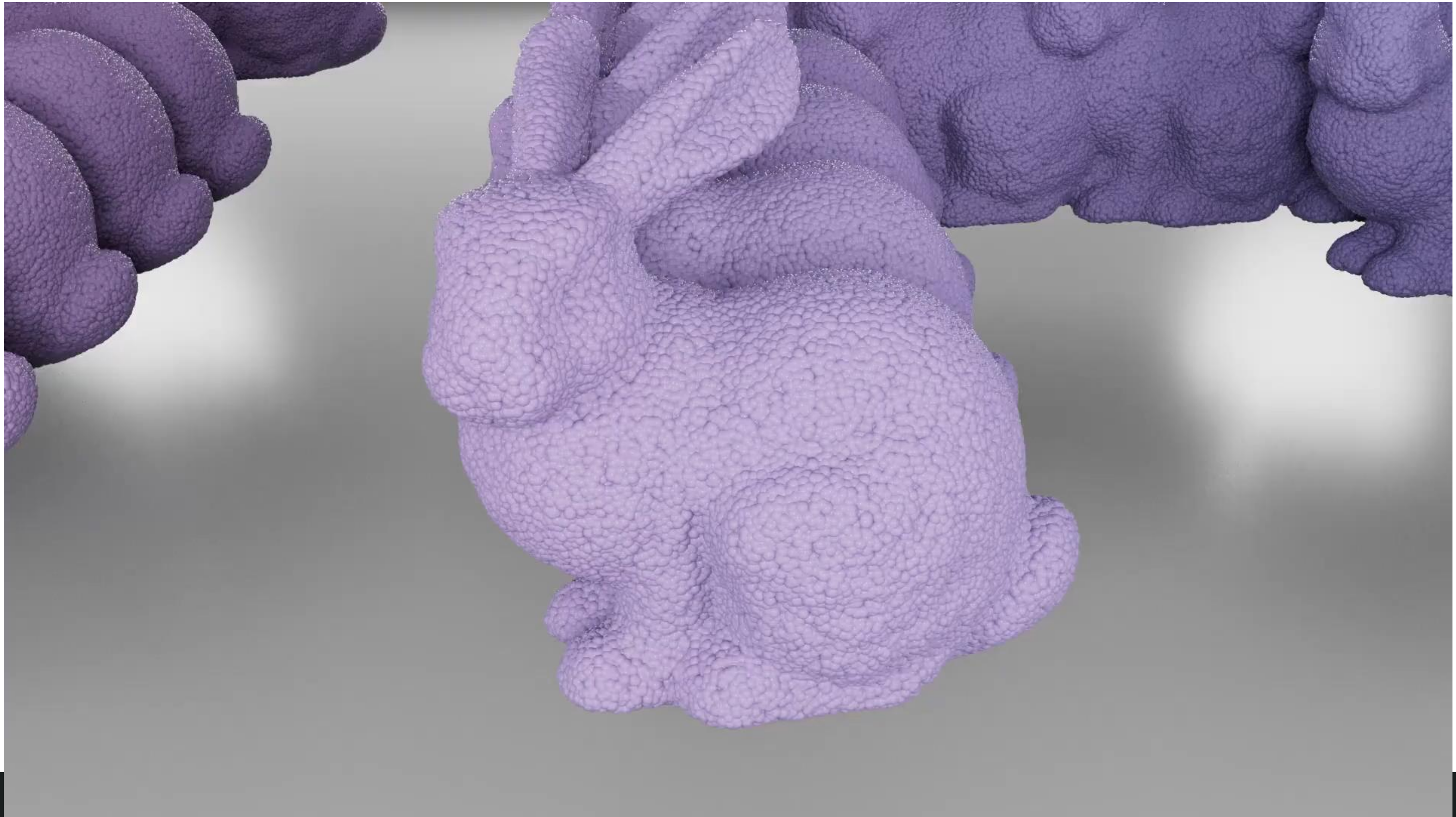
2.69X

2.45X 3.73X 14.62X

- Compared to Houdini, adaptive octree approach for both pressure Poisson and variational viscosity equation, per-step statistics

EXPERIMENTS AND RESULTS

SIGGRAPH 2022
VANCOUVER+ 8-11 AUG



SUMMARY

- Contributions
  - A matrix-free matrix vector multiplication implementation using SIMD and OpenVDB
  - A Galerkin coarsening strategy which preserves stencils
  - A parallel implementation of the coarsening strategy
  - A multi-color Gauss-Seidel smoother which is efficient for variational viscosity equation

SUMMARY

- Our Website
  - http://computationalsciences.org/publications/shao-2022-multigrid.html

Scan Here

Thank you for listening!